Eötvös Loránd University
Institute of Mathematics
Department of Analysis

Ph.D. thesis

Linear functional equations

Gergely Kiss

EpdupsbmTdippm; Nbuifnbujdt
Ejsfdups; Njlmpt Mbd{lpwjdi
nfncfs pg uif Ivohbsjbo Bdbefnz pg Tdjfodft

EpdupsbmQsphsbn; Qvsf Nbuifnbujdt
Ejsfdups; Boesbt T–vdt
Qspgfttps- Epdups pg Tdjfodft

Supervisor: Miklós Laczkovich
nfncfs pg uif Ivohbsjbo Bdbefnz pg Tdjfodft

Bqsjm3125

Contents

Acknowledgement

J xpvme mjlf up fyqsftt nz tjodfsf hsbujuvef up nz tvqfswjtps- Njlmpt Mbd-lpwjdi gps uif dpoujovpvt tvqqpsu bt xfmmbt gps uif jouf sftujoh sftfbsdi upqjdt0 J uijol xjui pvu ijt tvqqpsu uijt xpsl xpvme cf ofwfs ffiojtife boe nbz cf ofwfs tubsufe0 Ijt bewjdf boe fodpvsbhfnfou)cpui jo nbuifnbujdbmboe qfstpobmn buufst+nfbou b mpu up nf0 J dpvme opu ibwf jn bhjofe ibwjoh b cfuufs tvqfswjtps0

J bmtp xpvme mjlf up uibol up nz qbsfout gps uifjs dpotubou dbsf- mpwf boe tvqqpsu- up nz cspuifs boe tpnf pgnz gsjfoet gps uifjs fodpvsbhfnfou0

Mbtu cvu opu mfbtu- J xpvme mjlf up uibol up bmmn fn cfst pgEfqbsun fou pgTupdi btujd jo CNF xifsf J hpu mpu pgtvqqpsu gps nz xpsl0

1 Introduction

Mfu ℂ efopuf ui f fffm pg dpn qmfy ovn cfst0 Yf bsf dpodfsofe xjui ui f mjofbs gvodujpobm fr vbujpo

$$\prod_{i=1}^{n} a_i f)b_i x\ ,\ \ c_i y+?\ 1 \qquad)x,y\ /\ \mathbb{C}+; \qquad)2+$$

x i fsf a_i, b_i, c_i bsf hjwfo dpn qmfy ovn cfst- boe f ; ℂ ⇐ ℂ jt ui f vol opxo gvodujpo0 Cz b x fml opx o sftvmu pg M0T-fl fmi jej]47‘)tff Ui fpsfn 3037+ voefs tpn f n jme dpoejujpot po ui f fr vbujpo- fwfsz tpmvujpo pg)2+jt b hfofsbmj-fe qpmzopn jbm0)Gps ui f efffiojujpo pg hfofsbmj-fe qpmzopn jbmt tff ui f ofyu tfdujpo0+ Cvu ui f ffiofs tusvduvsf pg ui f tpmvujpot i bt cffo jowftujhbufe pomz sfdfoumz0 Evsjoh ui f qbtu dpvqmf pg zfbst tfwfsbm qbqfst i bwf cffo efwpufe up ui f qspcmfn pg ffiofejoh ui f hfofsbmj-fe qpmzopn jbm)jo qbsujdvmbs- beejujwf+ tpmvujpot pg tpn f tqfdjbmdbtft pg fr vbujpo)2+0 N ptu pg ui ftf sfdfou jowftujhbujpot x fsf tubsufe joefqfoefoumz cz B0 Wbshb boe ui f bvui ps pg ui f qsftfou ui ftjt bspvoe 311: 0 Tff]24‘-]49‘-]4: ‘-]53‘=tff bmtp ui f sfgfsfodft pg]49‘ boe]53‘0 Sftvmut dpodfsojoh ui f beejujwf tpmvujpot pg ui f hfofsbmfr vbujpo)2+dbo cf gpvoe jo]26‘-]51‘ boe]52‘0 Jo ui jt ui ftjt x f dpoujovf ui ftf jowftujhbujpot boe buufn qu up hjwf b dpn qmfuf eftdsjqujpo pg ui ptf tpmvujpot pg ui f fr vbujpot pg uzqf)2+x i jdi bsf hfofsbmj-fe qpmzopn jbmt0

Ui fsf bsf jn qpsubou tqfdjbmdbtft x i fo fwfsz tpmvujpo jt bvupn bujdbmmz b hfofsbmj-fe qpmzopn jbm0 Tvdi fr vbujpot bsf- f0h0-

$$\prod_{i=1}^{n} a_i f)b_i x\ ,\ \ y+?\ 1$$

boe

$$\prod_{i=1}^{n} a_i f)b_i x\ ,\ )2\ \ b_i+y+?\ 1.$$

Ui fsfgpsf- jo ui ftf tqfdjbmdbtft pvs sftvmut hjwf ui f dpn qmfuf eftdsjqujpo pg ui f tqbdf pg tpmvujpot- bu mfbtu jo qsjodjqmf0

Mfu S efopuf ui f tfu pg tpmvujpot efffiofe po ui f fffme hfofsbufe cz ui f qbsbn fufst b_i, c_i0 Ju jt dmfbs ui bu S jt b mjofbs tqbdf pwfs ℂ boe jt dmptfe voefs qpjouxjtf dpowfshfodf0 Jg S jt bmtp usbotmbujpo jowbsjbou- ui fo S jt b wbsjfuz0 Yf x bou up eftdsjcf S cz qsftfoujoh tpmvujpot pg tjn qmf tusvduvsf ui bu tqbo S0

Ui f tjuvbujpo jt ui bu pg *spectral analysis and synthesis*0 Jo gbdu- ui f n ptu jn qpsubou dpousjcvujpo pg pvs sftvmut up ui f ui fpsz pg mjofbs gvodujpobmfr vbujpot jt ui f bqqmjdbujpo pg tqfdusbmbobmztjt boe tzoui ftjt up tpn f wbsjfujft sfmbufe up ui f tqbdft pg tpmvujpot pg ui f fr vbujpot0 Ui f jefb pg bqqmzjoh tqfdusbmbobmztjt up ui ftf wbsjfujft ffistu bqqfbsfe jo]26‘0 Ui f n fui pe pg tqfdusbmtzoui ftjt x bt vtfe jo]24‘ boe]25‘0 Ui f n bjo ejgdvmuz jo bqqmzjoh tqfdusbmtzoui ftjt jo ui ftf wbsjfujft tufn t gspn ui f gbdu ui bu ui f voefsmzjoh hspvqt bsf pg joffojuf upstjpo gsff sbol- xifsf tqfdusbmtzouiftjt gbjmt jo tpnf wbsjfujft0 Yf pwfsdpnf uijt ejıdvmuz cz pctfswjoh uibu b npsf hfofsbmgpsn pg tqfdusbmtzouiftjt-dbmmfe mpdbmtqfdusbmtzouiftjt ipmet po fwfsz dpvoubcmf hspvq- boe uibu jo uif wbsjfujft jo rvftujpo uif qpmzopnjbmfyqpofoujbmt boe mpdbmqpmzopnjbmfyqpofoujbmt dpjodjef0

Jo pvs eftdsjqujpo pg tpmvujpot pg tjnqmf tusvduvsf efsjwbujpot boe fffm bvupnpsqijtnt qmbz b lfz spmf0 Uif pctfswbujpo uibu efsjwbujpot dbo cf vtfe jo uif eftdsjqujpo pg tpmvujpot xbt ffistu nbef jo]24‘0

Yf tibmmbqqmz uif hfofsbmsftvmut up tfwfsbmfrvbujpot xjui tqfdjbmqspqfsujft=f0h0 ibwjoh bmhfcsbjd qbsbnfufst fud0

Yf bmtp hjwf bo bqqmjdbujpo pg ejtdsfuf tqfdusbmtzouiftjt jo Tfdujpo 8- xifsf xf jo. uspevdf uif tp.dbmmfe ejtdsfuf Qpnqfjv qspcmfn- boe qsftfou b tpmvujpo up b tqfdjbmdbtf qptfe cz M0Qŏtb0

2 Preliminaries

2.1 Additive functions, polynomials, generalized polynomials, local polynomials

Mfu G boe G^∞cf Bcfrjbo hspvqt0 Ui f ejfifsfodf pqfsbups Λ_h jt efffbofe cz

$$\Lambda_h f)x+? \ f)x, \ h+ \ f)x+ \quad)x, h \ / \ G+$$

gps fwfsz f ; $G \Leftarrow G^\infty$0 Ui f pqfsbups Λ_h^n efopuft ui f dpn qptjujpo $\underbrace{\Lambda_h \ldots \Lambda_h}_{n}$0 Ju jt fbtz up di fdl ui bu $\Lambda_{h_1}\Lambda_{h_2}$? $\Lambda_{h_2}\Lambda_{h_1}$ gps fwfsz $h_1, h_2 \ / \ G$0

Cz bo beejujwf gvodujpo f ; $G \Leftarrow G^\infty$xf n fbo b gvodujpo tbujtgzjoh ui f gvodujpobm frvbujpo $f)x+, \ f)y+? \ f)x, \ y+$gps fwfsz $x, y \ / \ G$0 Jo pui fs xpset- f jt beejujwf jgju jt b i pn pn psqi jtn gspn G up G^∞0

B n bq F ; $G^i \Leftarrow G^\infty$jt dbmfe i.beejujwf jgju jt b i pn pn psqi jtn jo fbdi pgjut wbsjbcmft- ui f pui fs wbsjbcmft cfjoh ffiyfe0 Ui f gvodujpo F ; $G^i \Leftarrow G^\infty$jt tbje up cf tzn n fusjd- jgju ublft ui f tbn f wbmvf bu fwfsz qfsn vubujpo pgjut wbsjbcmft0

Ui f gvodujpo f jt b)hfofsbmj{fe+*monomial* pg efhsff i jg ui fsf fyjtut b tzn n fusjd- i.beejujwf n bq F ; $G^i \Leftarrow G^\infty$tvdi ui bu $f)x+? \ F)x, \ldots, x+$gps fwfsz $x \ / \ G$0 Gps fwfsz F ; $G^i \Leftarrow G^\infty$ ui f gvodujpo $f)x+? \ F)x, \ldots, x+)x \ / \ G+$jt tbje up cf ui f *diagonal* pg F- efopufe cz ejbh$)F+$0 Jg G^∞jt vojrvfmz ejwjtjcmf cz i"=ui bu jt- jgui f frvbujpo i" $\times x$? a i bt b vojrvf tpmvujpo gps fwfsz $a \ / \ G^\infty$ui fo ui f tzn n fusjd i.beejujwf gvodujpo F jt vojrvfmz efufsn jofe cz f0 Ui jt jt b dpotfrvfodf pgui f gpmmpx joh x fmm lopx o gbdu0

Lemma 2.1. *Let* $f)x+? \ F)\underbrace{x, \ldots, x}_{i}+$*be a monomial, where* F *is symmetric and* i*-additive. Then*

$$i\text{"} \times F)x_1, x_2, \ldots, x_i+? \ \Lambda_{x_1}\Lambda_{x_2} \ldots \Lambda_{x_i} f)t+ \qquad)3+$$

for every $t, x_1, x_2, \ldots, x_i \ / \ G$.

Proof. Tff]27- Mfn n b 260 03- q0 4: 5'0

Yf ti bmmvtf ui f mfn n b jo ui f tqfdjbmdbtf x i fo G^∞? $\mathbb{C}$0 Yf hfu ui bu b gvodujpo f ; $G \Leftarrow \mathbb{C}$ jt b hfofsbmj{fe n popn jbmpgefhsff i jgboe pomz jgui f gvodujpo F efffbofe cz)3+jt i.beejujwf- boe $f)x+? \ F)x, \ldots, x+$gps fwfsz $x \ / \ G$0

B gvodujpo f jt dbmfe b *generalized polynomial* jg f ? $\prod_{i=0}^{n} f_i$- x i fsf f_i jt b n popn jbm pgefhsff i)i ? 2, $\ldots, n$+boe f_0 jt b dpotubou0

Ui f gpmmpx joh ui fpsfn jt b dmbttjdbmsftvmu pg N b{vs boe Psmjd-]34'0

Theorem 2.2. *Let* f ; $\mathbb{C} \Leftarrow \mathbb{C}$ *be a function and* $n \ / \ \mathbb{N}$. *Then the following are equivalent:*

)j+

$$\Delta_{h_1}\dots\Delta_{h_{n+1}} f)x+?\ 1$$

for every $h_1,\dots,h_{n+1},x\ /\ \mathbb{C}$.

)jj+

$$\Delta_h^{n+1} f)x+?\ 1$$

for every $h\ /\ \mathbb{C}$.

)jjj+

$$f\ ?\ \prod_{i=0}^{n} f_i$$

where f_i *is a monomial of degree* i *(*$i\ ?\ 2,\dots,n$*) and* f_0 *is a constant, thus* f *is a generalized polynomial.*

Jo]28' Mbd-l pwjdi tuvejft ui ftf qspqfsujft jo b n psf hfofsbmdpoufyu0 Gps ui f tbl f pg dpn qmfufoftt xf sfdbmmtpn f pg ui f efffiojujpot boe sftvmt pg]28'0

Mfu f cf b n bq gspn G up G'0 Jo]28' ui f gpmpx joh qspqfsujft pg f bsf efffiofe0

P)n+; $f\ ?\ \prod_{i=0}^{n} f_i$- x i fsf f_i jt b n popn jbmpgefhsff i)$i\ ?\ 2,\dots,n$+boe f_0 jt dpotubou0

P$_{\text{loc}}$)n+; Gps fwfsz $g_1,\dots,g_s\ /\ G$ ui fsf bsf frfm fouc$_i$ $/\ G^{\infty}$)$i\ ?\ $)$i_1,\dots,i_s$+;$i_1,\dots,i_s \subset$ 1, \i\? i_1 , ××× $i_s \geq n$+tvdi ui bu

$$f)k_1 g_1\ ,\ \ \times\times\times\ \ k_s g_s+?\ \prod_{i\ \subset n} k_1^{i_1}\times\times\times k_s^{i_s} c_i$$

gps fwfsz $k_1,\dots,k_s\ /\ \mathbb{Z}$0

P$_{\text{lin}}$)n+; Gps fwfsz $a,b\ /\ G$ ui fsf bsf frfm fout $c_i\ /\ G$')$i\ ?\ 1,\dots,n$+tvdi ui bu

$$f)a\ ,\ \ kb+?\ \prod_{i=0}^{n} k^i \times c_i$$

gps fwfsz $k\ /\ \mathbb{Z}$0

MD)n+; $\Delta_{h_1}\dots\Delta_{h_{n+1}} f)x+?\ 1$ gps fwfsz $h_1,\dots,h_{n+1},x\ /\ G$0

ID)n+; $\Delta_h^{n+1} f)x+?\ 1$ gps fwfsz $h,x\ /\ G$0

R)n+; Ui fsf bsf gvodujpot $f_1,\dots,f_{n+1}$; $G \Leftarrow G^{\infty}$boe joufhfst a_i boe b_i tvdi ui bu b_i \? 1)$i\ ?\ 2,\dots,n\ ,\ 2$+ boe

$$f)x+?\ \prod_{i=1}^{n+1} f_i)a_i x\ ,\ \ b_i y+$$

gps fwfsz $x,y\ /\ G$.

Qspqptjujpo 2 pg]28' tubuft ui f gpmpx joh0

Proposition 2.3. *For every function* $f\,;G \Leftarrow G^{\infty}$ *we have*

$$P)n+\Leftrightarrow P_{loc})n+\Leftrightarrow MD)n+\Leftrightarrow ID)n+\Leftrightarrow R)n+;$$

and

$$P_{loc})n+\Leftrightarrow P_{lin})n+\Leftrightarrow ID)n+.$$

Jo]28‘ ju jt brtp ti px o ui bu ui f sfwfstf jn qrjdbujpot ep opu i pm jo hfofsbm0 I pxfwfs-jo b rbshf drbtt pg tqfdjbmdbtft- x i fo G jt ejwjtjcrfi- bmpg ui f n bsf frvjwbrfou0

Gspn ui jt qpjou po x f ti bmpom dpotjefs gvodujpot n bqqjoh up ui f fffrm pg dpn qrfiy ovn cfst ℂ0

Theorem 2.4. *If the group* G^{∞} *is the additive group of* ℂ*, then we have*

$$P)n+\Rightarrow\Leftrightarrow P_{loc})n+\Rightarrow\Leftrightarrow P_{lin})n+\Rightarrow\Leftrightarrow MD)n+\Rightarrow\Leftrightarrow ID)n+.$$

Joeffe- [0Eklpl pwjd]9‘ qspwfe $MD)n+?\Leftrightarrow P)n+$ bttvn joh ui bu G^{∞} jt vojrvfm ejwjtjcrfi cz)n , 2+"0)Bopui fs qsppg tvqqptjoh ui bu G, G^{∞} bsf ejwjtjcrfi boe G^{∞} jt upstjpo.gsff x bt hjwfo cz M0T-fl f mi jej jo]42‘0+ Jo qbsujdvrbs- $MD)n+?\Leftrightarrow P)n+$ i prnt jg $G^{\infty}?$ ℂ0 Ui vt-cz Qspqptjujpo 304-

$$P)n+\Rightarrow\Leftrightarrow P_{loc})n+\Rightarrow\Leftrightarrow MD)n+.$$

Ui f frvjwbrfodf pg $MD)n+$ boe $ID)n+$ jo ui f dbtf pg $G^{\infty}?$ ℂ gpmpx t gspn]28- Ui fpsfn 25‘- boe ui jt jn qrjft ui f frvjwbrfodf pg ui f sfn bjojoh dpoejujpot0

Ui fsfgpsf- *a function* $f\,;G \Leftarrow$ ℂ *is a generalized polynomial if and only if it satisfies any of the conditions* $P)n+,\ P_{loc})n+,\ P_{lin})n+,\ MD)n+,\ ID)n+.$ Ui f tn bmftu n gps x i jdi f tbujtffft boz pg ui ftf dpoejujpot jt dbmfe ui f efhsff pg f. Ui f efhsff pg ui f jefoujdbmz –fsp gvodujpo jt- cz efffojujpo- 20

Ju jt fbtz up tff ui bu ui f hfofsbmj-fe qpmzopn jbmt pg efhsff 1 bsf ui f opo–fsp dpotubou gvodujpot0 Ui f hfofsbmj-fe qpmzopn jbmt pg efhsff 2 bsf pg ui f gpsn $a)x+,\ b$ x i fsf $a\,;G \Leftarrow$ ℂ jt b opo–fsp beejujwf gvodujpo boe b jt dpotubou0

Proposition 2.5. *The set of all generalized polynomials is an algebra over* ℂ.

Proof. Y f qspwf ui bu jg f boe g bsf hfofsbmj-fe qpmzopn jbmt pg efhsff $\geq n$- ui fo $c\times f,\ f,\ g$ boe $f\times g$ bsf hfofsbmj-fe qpmzopn jbmt0 Ui jt fbtjmz gpmpx t cz joevdujpo po n- vtjoh ui f gpsn vrbt

$$\Lambda_h)cf+?\ c\Lambda_h f,$$

$$\Lambda_h)f\ ,\ \ g+?\ \Lambda_h f\ ,\ \ \Lambda_h g,$$

$$\Lambda_h)f \times g+? \ \Lambda_h f \times \Lambda_h g \ , \ f \times \Lambda_h g \ , \ \Lambda_h f \times g.$$

□

B gvodujpo f ; $G \Leftarrow \mathbb{C}$ jt b *polynomial* jg ju cfmpoht up uif bmhfcsb hfofsbufe cz uif beejujwf gvodujpot boe uif dpotubou gvodujpot0

Proposition 2.6. *Let G be an Abelian group. Then every polynomial function f ; $G \Leftarrow \mathbb{C}$ is a generalized polynomial.*

Proof. Mfu q cf b qpmzopnjbm0 Uifo ju jt pg uif gpsn $Q)a_1)x+, \ldots, a_n)x++$ xifsf Q / $\mathbb{C}]x_1, \ldots, x_n$‘ boe $a_1, \ldots, a_n$ bsf beejujwf gvodujpot0 Uivt q jt jo uif bmhfcsb hfofsbufe cz uif hfofsbmj{fe qpmzopnjbmt $a_1, \ldots, a_n$ boe ju gpmmpxt uibu q jt b hfofsbmj{fe qpmzopnjbm0

□

Uif sfwfstf jnqmjdbujpo jt opu usvf jo hfofsbm0 T{fl fmzi jej]46‘ dpotusvdufe uif gpmmpxjoh dpvoufsfybnqmf0

Mfu F^ω efopuf uif gsff Bcfmjbo hspvq hfofsbufe cz dpvoubcmz jogjojufmz nboz hfofsbupst0 Yf tibmm sfqsftfou F^ω bt

$$F^\omega \ ? \ \})x_1, x_2, \ldots+; x_k \ / \ \mathbb{Z} \ )k \ ? \ 2, 3, \ldots, +\emptyset i_0, x_i \ ? \ 1 \ )i > i_0+|,$$

xifsf uif tvn pg uif fmfnfout $)x_1, x_2, , \ldots, +$boe $)y_1, y_2, , \ldots, +$jt $)x_1 \ , \ y_1, x_2 \ , \ y_2, , \ldots, +$.

Yf efopuf cz $C)G+$uif mjofbs tqbdf)pwfs $\mathbb{C}$+pg bmm gvodujpot nbqqjoh G joup $\mathbb{C}$0

Proposition 2.7. *The translates of a polynomial function generates a finite dimensional linear subspace of $C)G+$.*

Proof. Ju jt fbtz up tff uibu uif usbotmbuft pg b opo.{fsp beejujwf gvodujpo hfofsbuf b mjofbs tqbdf pg ejnfotjpo uxp0 Po uif puifs iboe- jg uif usbotmbuft pg f boe h hfofsbuf bo n boe bo m ejnfotjpobm mjofbs tvctqbdf pg $C)G+$ sftqfdujwfmz- uifo uif usbotmbuft pg f , h hfofsbuf bo bu nptu n , m ejnfotjpobm tqbdf- boe uif usbotmbuft pg $f \times h$ hfofsbuf bo bu nptu $n \times m$ ejnfotjpobm tqbdf0 Uif tubufnfou pg uif Qspqptjujpo dmfbsmz gpmmpxt gspn uiftf pctfswbujpot0

□

Theorem 2.8)T{fl fmzi jej]46‘+. *There is a generalized polynomial g ; $F^\omega \Leftarrow \mathbb{C}$ which is not a polynomial.*

Proof. Mfu

$$g)x+? \ \prod_{i=1}^{\in} x_i^2 \qquad)\exists x \ ? \ )x_i+_{i=1}^{\in} \ / \ F^\omega+.$$

Uif gvodujpo $g)x+$jt b hfofsbmj{fe qpmzopnjbm tjodf $B)x, y+? \ \prod_{i=1}^{\in} x_i y_i$ jt b cjbeejujwf gvodujpo boe ejbh$)B+)x+? \ g)x+$ipmet0 Tvqqptf g jt b qpmzopnjbm0 Uifo- cz Qspqptjujpo 308- uif usbotmbuft pg g hfofsbuf b gjojuf ejnfotjpobm mjofbs tvctqbdf pg $C)G+$0

Uif gvodujpot pg uif gpsn $x \forall\!\!\Leftarrow B)x, y+)x$ / G+bsf beejujwf gps fwfsz y / G0 Jg $y_{(k)}$?)1, . . . , 1, 2, 1 . . .+ xifsf uif 2 bqqfbst bu uif k^{th} dppsejobuf- uifo $B)x, y_{(k)}+$? x_k- uif k^{th} dppsejobuf pg x0 Ju jt dmfbs uibu uif nbqt $x \forall\!\!\Leftarrow x_k$)k ? 2, 3, . . .+bsf mjofbsmz joefqfoefou pwfs ℂ0 Tjodf 3 ×$B)x, y+$? $g)x$, $y+$ $g)x+$ ju gpmmpxt uibu uif gvodujpot $x \forall\!\!\Leftarrow x_k$ cfmpoh up uif mjofbs tqbdf hfofsbufe cz uif usbotmbuft pg g0 Uijt jnqmjft uibu uif usbotmbuft pg g hfofsbuf bo joggjojuf ejnfotjpobm tqbdf0 Uifsfgpsf- bt xf tbx bcpwf- g dboopu cf b qpmzopnjbm0

□

Cz uif upstjpo gsff sbol pg G xf nfbo uif dbsejobmjuz pg b nbyjnbm joefqfoefou tztufn pg fmfnfout pg joggjojuf psefs0 Jo puifs xpset uif upstjpo gsff sbol pg G jt uif nbyjnbm dbsejobmjuz κ tvdi uibu G dpoubjot uif gsff Bcfmjbo hspvq pg sbol κ bt b tvchspvq0

Uif tubufnfou pg Uifpsfn 30: jn nfejbufmz jnqmjft uif gpmmpxjoh uifpsfn]46- Uifpsfn 300

Theorem 2.9. *For every Abelian group G with infinite torsion free rank there is a generalized polynomial f ; $G \Leftarrow$ ℂ which is not a polynomial.*

Proof. Joeffe- bo Bcfmjbo hspvq G xjui joggjojuf upstjpo gsff sbol dpoubjot b tvchspvq H xijdi jt jtpnpsqijd up F^{ω}0 Uif qsppg pg Uifpsfn 30: hjwft b tznnfusjd cjbeejujwf gvodujpo B ;)H ∗ H+⇐ ℂ tvdi uibu g ? ejbh)B+jt opu b qpmzopnjbm po H0 Tjodf ℂ jt ejwjtjcmf- ju gpmmpxt gspn b xfmm lopxo gbdu pg hspvq uifpsz uibu B dbo cf fyufoefe up G ∗ G bt b tznnfusjd cjbeejujwf gvodujpo0

Uifo $B)x, x+$jt b hfofsbmj{fe qpmzopnjbm po G xijdi jt opu b qpmzopnjbm tjodf jut sftusjdujpo up H jt opu b qpmzopnjbm0

□

Po uif puifs iboe- ju jt fbtz up qspwf uif gpmmpxjoh uifpsfn 0

Theorem 2.10)N0 Mbdl-l pwjdi]29‘+. *If G is a finitely generated Abelian group and f ; $G \Leftarrow$ ℂ is a generalized polynomial, then f is a polynomial.*

Proof. Fwfsz ggojufmz hfofsbufe Bcfmjbo hspvq jt jtpnpsqijd up $\mathbb{Z}^k$ ∗ H gps b tvjubcmf k- xifsf H jt b ggojuf Bcfmjbo hspvq0 Uivt- ju jt fopvhi up efbm xjui uif gpmmpxjoh uxp tqfdjbm dbtft0

Po uif hspvq $\mathbb{Z}^k$ fwfsz hfofsbmj{fe qpmzopnjbm jt b qpmzopnjbm0 Joeffe- uif qspqfsujft $P)n+$boe $P_{loc})n+$bsf frvjwbmfou0 Uivt uif hfofsbmj{fe qpmzopnjbm f jt b sftusjdujpo pg b qpmzopnjbm p / ℂ]$x_1, \ldots, x_k$‘ up $\mathbb{Z}^k$0 Tjodf uif qspkfdujpo poup boz dppsejobuf π_i ; $x \forall\!\!\Leftarrow x_i$ jt beejujwf gps fwfsz i ? 2, . . . , k- uifsfgpsf $f)x+$? $P)\pi_1)x+, \ldots, \pi_k)x+$0

Jg H jt b ffiojuf Bcfmjbo hspvq- ui fo fwfsz hfofsbmj–fe qpmzopnjbmjt dpotubou po H0 Joeffe- po H fwfsz beejujwf gvodujpo jt –fsp0 Ui vt fwfsz k.beejujwf gvodujpo jt –fsp bt xfmmboe ui vt fwfsz hfofsbmj–fe qpmzopnjbmjt dpotubou po H0

Jg $)x,y+\not\Leftarrow f)x,y+))x,y+ / \mathbb{Z}^k * H+$jt b hfofsbmj–fe qpmzopnjbmui fo $y \not\Leftarrow f)x,y+$ $)y / H+$jt b hfofsbmj–fe qpmzopnjbmpo H gps fwfsz ffiyfe $x / \mathbb{Z}^k$0 Ui vt $f)x,y+? f)x,1+$ gps fwfsz $x / \mathbb{Z}^k$0 Tjodf $f)x,1+$jt b hfofsbmj–fe qpmzopnjbmpo $\mathbb{Z}^k$- ju jt b qpmzopnjbm0 $\square$

Opx xf ublf pof tufq gvsui fs cz hfofsbmj–joh ui f opubujpo pg qpmzopnjbmt0 Y f tbz ui bu ui f gvodujpo $f ; G \Leftarrow \mathbb{C}$ jt b *local polynomial* jg- gps fwfsz ffiojufmz hfofsbufe tvchspvq H pg G- ui f sftusjdujpo $f\backslash_H$ jt b qpmzopnjbmpo H0 Qspqfsuz $P_{loc})n+$hvbsbouff t ui bu fwfsz hfofsbmj–fe qpmzopnjbmjt b mpdbmqpmzopnjbm0

Ui f gpmmpxjoh fybnqmf ti pxt ui bu ui f sfwfstf jn qmjdbujpo jt opu ofdfttbsz usvf0

Proposition 2.11)N0Mbd–lpwjdi]29‘+ *There is an $f ; F^\omega \Leftarrow \mathbb{C}$ which is a local polynomial and is not a generalized polynomial.*

Proof. Mfu

$$P)x+? \prod_{i=1}^{\in} x_i^i$$

gps fwfsz $x ?)x_1, x_2, \ldots + / F^\omega$. Ju jt fbtz up di fdl ui bu $P)x+$jt b mpdbmqpmzopnjbm0

Ui f fmfn fout $)x_1, x_2, \ldots +$xjui $x_k ? 1$ gps fwfsz $k > n$ dpotujuvuf b tvchspvq H_n pg F^ω0 Ju jt fbtz up tff ui bu ui f sftusjdujpo pg $P)x+$up H_n jt b hfofsbmj–fe qpmzopnjbmpg efhsff n0 Ui fsfgpsf- $\Lambda_{h_1} \ldots \Lambda_{h_n} P)x+\not? 1$ gps tpn f $h_1, \ldots, h_n, x / H_n$0 Ui jt jn qmjft ui bu $P)x+$jt opu b hfofsbmj–fe qpmzopnjbmpg efhsff $\geq n$0 Tjodf ui jt jt usvf gps fwfsz n- $P)x+$jt opu b hfofsbmj–fe qpmzopnjbm0 $\square$

Bt bcpwf- ju jt fbtz up hfofsbmj–f ui jt sftvmu jo ui f gpmmpxjoh xbz0

Theorem 2.12. *For every Abelian group G with infinite torsion free rank there is a local polynomial $g ; G \Leftarrow \mathbb{C}$ which is not a generalized polynomial.*

Ju jt dmfbs ui bu jg G jt b ffiojufmz hfofsbufe Bcfmjbo hspvq- ui fo fwfsz mpdbmqpmzopnjbm po G jt b qpmzopnjbm0 Dpn qbsjoh xjui Ui fpsfn 3021- xf pcubjo ui f gpmmpxjoh0

Proposition 2.13. *If G is a finitely generated Abelian group then, for every $f ; G \Leftarrow \mathbb{C}$ we have*

$$f \text{ is a polynomial} \infty f \text{ is a generalized polynomial} \infty f \text{ is a local polynomial.}$$

2.2 Discrete spectral analysis and spectral synthesis

MfuG cf b mpdbmmz dpn qbdu hspvq boe mfuC)G+efopuf ui f mjofbs tqbdf pgbmmdpn qmfy wbmvfe gvodujpot efgjofe po G frvjqqfe xjui ui f upqpmphz pg vojgpsn dpowfshfodf po dpn qbdu tfut0 Jo tfwfsbmejggfsfoubsfbt pgn bui fn bujdt)ejggfsfoujbmboe ejggfsfodf frvbujpot- ui fpsz pghspvq sfqsftfoubujpo- i bsn pojd bobmztjt- fud0+tpn f tqfdjbmdmbttft pgtvctqbdft pgC)G+ qmbz gvoebn foubmspmft0 Pof pgui fn jt ui f dmbtt pgusbotmbujpo jowbsjbou- dmptfe tvctqbdft pgC)G+0 Tqfdusbmbobmztjt boe tzoui ftjt efbmxjui ui f eftdsjqujpo pgusbotmbujpo jowbsjbou gvodujpot tqbdft pwfs mpdbmmz dpn qbdu hspvqt0 Jo psefs up tvn n bsj{f ui f n bjo n pujwbujpo pgui jt tfdujpo xf rvpuf gspn ui f bsujdmf]46';

'Ui f gvoebn foubmqspcmfn jt up ejtdpwfs ui f tusvduvsf pgtvdi tqbdf pggvodujpot- ps n psf fybdumz- up gjoe bo bqqspqsjbuf dmbtt pgcbtjd gvodujpot- ui f cvjmejoh cmpdl t- xi jdi tfswf bt 'uzqjdbmfmfn fout' pgui f tqbdf- b l joe pgcbtjt0

Jo ui f tfrvfmxf pomz dpotjefs Bcfmjbo hspvqt xjui ui f ejtdsfuf upqpmphz0 Ui fo C)G+ jt ui f tbn f bt $\mathbb{C}^G$- ui f mjofbs tqbdf pgbmmdpn qmfy wbmvfe gvodujpot efgjofe po G0 Yf frvjq C)G+? $\mathbb{C}^G$ xjui ui f qspevdu upqpmphz0

B opo{fsp gvodujpo m / C)G+jt dbmmfe bo *exponential* jg$m$ jt n vmujqmjdbujwf=ui bu jt- jg$m)x$, $y+?$ $m)x+\times m)y+$gps fwfsz x, y / G0 Bo *exponential monomial* jt ui f qspevdu pg b qpmzopn jbmboe bo fyqpofoujbmb *polynomial-exponential function* jt b gjojuf tvn pg fyqpofoujbmn popn jbmt0

Ui f usbotmbujpo pqfsbups T_h jt efgjofe cz $T_hf)x+?$ $f)x$, $h+)x$ / G+gps fwfsz h / G boe f / C)G+0 Yf dbmm$T_hf)x+)h, x$ / G+b usbotmbuf pg f0 Ui f gpmmpxjoh ui fpsfn]21- 2:- 45' hjwft b di bsbdufsj{bujpo pgqpmzopn jbmfyqpofoujbmgvodujpot0

Theorem 2.14. *Let G be a topological group (as a special case, locally compact). The translates of f ; G ⇐ ℂ generate a finite dimensional space if and only if f is a polynomial-exponential function.*

Opuf ui bu ui f usbotmbuft pgbo fyqpofoujbmgvodujpo hfofsbuf b pof ejn fotjpobmtvctqbdf pgC)G+0 Ui vt ui f bshvn fou vtfe jo ui f qsppg pgQspqptjujpo 308 hjwft ui f ajg(qbsu pgui f ui fpsfn 0

Yf ti bmmgfrvfoumz vtf ui f gpmmpxjoh xfmml opxo sftvmu0

Lemma 2.15. *Let $)G, \pm+$be an Abelian group, V be a translation invariant linear subspace of C)G+, and let $\prod_{i=1}^{M} p_i \times m_i$ / V, where $p_1, \ldots, p_M$ are nonzero generalized polynomials and $m_1, \ldots, m_i$ are distinct nonzero exponentials on G. Then $)\Lambda_{h_1} \ldots \Lambda_{h_k} p_i + \times m_i$ / V for every i and for every $h_1, \ldots, h_k$ / G.*

If p_i is not constant, then there is a nonzero additive function A ; G ⇐ ℂ such that $A \times m_i$ / V and m_i / V.

Proof. Tjodf m_i jt b opo–fsp fyqpofoujbm $m_i)x+$? 1 gps boz x / G0 Jg $p)x+$ $m)x+$ / V- ui fo $p)x \pm h+$ $m)x \pm h+$ boe c $p)x+$ $m)x+$ jt jo V gps boz c ? 1 / $\mathbb{C}$- cf dbvtf V jt b usbotmbujpo jowbsjbou mjofbs tqbdf0 Ui vt- ui f frvbujpo

$$p)x \pm h+ m)x \pm h+ \quad m)h+p)x+ m)x+ ?$$
$$? \;)p)x \pm h+ \quad p)x++m)x+ m)h+ ? \quad)\Lambda_h p)x++ m)x+ m)h+$$

ti px t ui bu $\Lambda_h p)x+$ $m)x+$ / V0 Ui f jufsbujpo pgui jt qspdftt qspwft ui f ffistutubufn foupgui f mfn n b0 Ju jt x fmm lopx o ui bu gps fwfsz hfofsbmj-fe qpmzopn jbm p ui fsf fyjtu bo joufhfs k boe bo beejujpobm gvodujpo A tvdi ui bu $\Lambda_{h_1} \ldots \Lambda_{h_k} p$ A jt dpotubou- boe $\Lambda_{h_1} \ldots \Lambda_{h_k} \Lambda_{h_{k+1}} p$ jt b opo–fsp dpotubou gps tpn f $h_1, \ldots, h_k, h_{k+1}$ / G0 Ui vt- jg p m / V- ui fo A m boe m bsf jo V- x i jdi dpn qmfuft ui f qsppg0 $\square$

Cz b *variety* x f n fbo b usbotmbujpo jowbsjbou dmptfe tvctqbdf pg $C)G+$0

Jg b wbsjfuz dpoubjot bo fyqpofoujbm gvodujpo- ui fo x f tbz ui bu *spectral analysis holds in this variety*0 Jg tqfdusbm bobmztjt i pmet jo fwfsz wbsjfuz po G- ui fo x f tbz ui bu *spectral analysis holds on* G.

Jg b wbsjfuz V jt tqboofe cz fyqpofoujbm n popn jbmt cfmpohjoh up V- ui fo x f tbz ui bu *spectral synthesis holds in variety* V0 Jg tqfdusbm tzoui ftjt i pmet jo fwfsz wbsjfuz po G- ui fo x f tbz ui bu *spectral synthesis holds on* G. Ju jt dmfbs ui bu jg tqfdusbm tzoui ftjt i pmet jo b wbsjfuz V ui fo tqfdusbm bobmztjt i pmet po V- bt x fmm0

Gps fybn qmf- tqfdusbm tzoui ftjt i pmet po $\mathbb{Z}$ cz b dmbttjdbm sftvmu po tfrvfodft tbujtgzjoh b mjofbs sfdvstjpo0 Joeffe- tvqqptf ui bu V jt b qspqfs wbsjfuz pg $C)\mathbb{Z}+$ n fbojoh ui bu V ? $C)\mathbb{Z}+$0 Bt V jt dmptfe- ju gpmmpx t ui bu ui fsf jt b k tvdi ui bu L ? $\}f|_{\{0,\ldots,k-1\}} \; ; f$ / $V|$ jt b qspqfs mjofbs tvctqbdf pg $\mathbb{C}^k$? $\mathbb{C}^{\{0,\ldots,k-1\}}$0 Mfu $)c_0, \ldots, c_{k-1}+$ cf b opo–fsp fmfn fou pg $\mathbb{C}^k$ qfsqfoejdvmbs up L0 Ui fo pof dbo ti px - vtjoh usbotmbujpo jowbsjbodf pg V ui bu

$$\prod_{i=0}^{k-1} c_i f)n \quad i+? \; 1 \qquad)4+$$

gps fwfsz n0 Ui fsfgpsf- ui f tfrvfodf $f)n+$ tbujtfift b mjofbs sfdvstjpo0 Mfu $p)x+?$ $\prod_{i=0}^{k-1} c_i x^i$ efopuf ui f di bsbdufsjtujd qpmzopn jbm pg ui f mjofbs sfdvstjpo tbujtgzjoh)4+ boe mfu λ_j)2 $\geq j \geq l$+ efopuf ui f sppu t pg $p)x+$ x jui n vmujqmjdjuz m_j $\subset$ 20 Ui fo $\prod_{j=1}^{l} m_j$? k0 Yf n bz bttvn f ui bu λ_j ? 1 x i jdi jt frvjwbmfou up c_0 ? 10 Ju jt x fmm lopx o ui bu fwfsz tpmvujpo f pg)4+ jt pg ui f gpsn

$$f)n+? \prod_{j=1}^{l} p_j)n+\lambda_j^n,$$

x i fsf p_j / $\mathbb{C}]x'$ jt b qpmzopn jbm boe efh $p_j \geq m_j$ gps fwfsz j0 Ju jt fbtz up di fdl ui bu- gps fwfsz j- ui f n bq n λ_j^n jt bo fyqpofoujbm gvodujpo- boe ui bu $p_j)n+$ jt b qpmzopn jbm gvodujpo po $\mathbb{Z}$0 Ui vt fwfsz tpmvujpo jt b mjofbs dpn cjobujpo pg qpmzopn jbmfyqpofoujbm gvodujpot- ui vt tqfdusbmtzoui ftjt i pmet po $\mathbb{Z}$0

Ui f gpmmpx joh ui fpsfn jt b hfofsbmj{bujpo pg ui f dbtf bcpwf boe ffistu xbt qspwfe cz N0 Mfgsbod]33‘0

Theorem 2.16. *For every finite n, spectral synthesis holds in every variety on the group $\mathbb{Z}^n$ equipped with the discrete topology.*

Ui f qsppgpgui jt ui fpsfn jt cbtfe po Lsvmm(t ui fpsfn boe pui fs sjoh ui fpsfujdbmsftvmut0 Cz ui f gpmmpx joh qspqptjujpo- xf pcubjo ui f tbn f sftvmu gps boz ffiojufm hfofsbufe Bcfmjbo hspvq- tjodf boz ffiojufm hfofsbufe Bcfmjbo hspvq jt b i pn pn psqi jd jn bhf pg $\mathbb{Z}^n$ gps tpn f n0

Proposition 2.17. *If spectral synthesis holds on an Abelian group G, then it holds for any homomorphic image of G.*

Ui f qsppg dbo cf gpvoe jo]47- q0 32‘0

Yf efopuf cz $r_0)G+$ ui f upstjpo gsff sbol pg G0 Ui f gpmmpx joh ui fpsfn jt ui f n bjo sftvmu pg]31‘0

Theorem 2.18. *Spectral analysis holds on a discrete Abelian group G if and only if $r_0)G+< 3^\omega$.*

S0 K0 Fmmjpu dmbjn fe jo]: ‘ ui bu tqfdusbmtzoui ftjt i pmet po fwfsz Bcfmjbo hspvq0 Fmmjpu(t qsppg- i px fwfs- xbt efgfdujwf0 Jo gbdu- ui f tubufn fou jt gbmtf- bt ui f gpmmpx joh ui fpsfn ti px t0

Theorem 2.19)M0 T{flfmzi jej]46‘+. *Spectral synthesis does not hold on any discrete Abelian group G with $r_0)G+\subset \omega$.*

Ui f qsppg jt cbtfe po ui f hfofsbmj{fe qpmzopn jbm $P)x+? \prod_{i=0}^{\in} x_i^2 \ )x \ / \ F^\omega+$ ui bu xbt dpotusvdufe jo Ui fpsfn 30 0 Mfu V efopuf ui f wbsjfuz hfofsbufe cz P0 Yf qspwf jo Ui fpsfn 30 ui bu P jt opu b qpmzopn jbm0 Po ui f pui fs i boe- ju jt fbtz up ti px ui bu fwfsz fmfn fou pg ui f wbsjfuz V jt pg ui f gpsn $cP)x+, \ a)x+, \ b$- xi fsf c, b bsf dpotubou boe a jt beejujwf0 Ui jt jn qmjft ui bu ui f pomz fyqpofoujbmjo V jt ui f jefoujdbmmz 2 gvodujpo- boe fwfsz qpmzopn jbmjo V jt pg ui f gpsn $a)x+, \ b$0 Ui fsfgpsf- ui f gvodujpo P jt opu jo ui f dmptvsf pg qpmzopn jbmfyqpofoujbmgvodujpot pg V0 Joeffe- fwfsz qpmzopn jbmjo V wbojti ft voefs ui f pqfsbujpo Λ_h^2 cvu P epft opu0

Po ui f pui fs i boe Mbd{lpwjdi boe T{flfmzi jej qspwf ui f gpmmpx joh vtfgvmsftvmu jo]32‘0

Theorem 2.20. *Spectral synthesis holds on a discrete Abelian group G if and only if $r_0)G+$ is finite.*

Yf ti bmtbz ui bu ui f gvodujpo f jt b *generalized polynomial-exponential*- jg f ? $\prod_{i=1}^n p_i \times m_i$- xi fsf $p_1, \ldots, p_n$ bsf hfofsbmj-fe qpmzopnjbmt boe $m_1, \ldots, m_n$ bsf fyqpofoujbmt0 Jg b wbsjfuz V jt tqboofe cz hfofsbmj-fe qpmzopnjbmfyqpofoujbmt cfmpohjoh up V- ui fo xf tbz ui bu *generalized spectral synthesis holds in the variety* V0

Tjodf fwfsz gvodujpo jo ui f wbsjfuz V pg ui f qsfwjpvt fybnqmf jt b hfofsbmj-fe qpmz. opnjbm ju gpmmpxt ui bu hfofsbmj-fe tqfdusbm tzoui ftjt i pmet jo V0 I pxfwfs- hfofsbmj-fe tqfdusbm tzoui ftjt epft opu i pme po boz Bcfmjbo hspvq G gps xi jdi $r_0)G+ \subset \omega$0 Ui jt jt b dpotfrvfodf pg ui f gpmmpxjoh ui fpsfn 0

Theorem 2.21)N 0 Mbd-l pwjdi]29‘+. *Let $Q)x+ ? \prod_{i=1}^{\in} x_i^i$ for every x ? $)x_1, x_2, \ldots, +/ F^\omega$, and let W denote the variety generated by Q. Then the set of generalized polynomial-exponentials contained in W is not dense in W.*

Yf ti bmtbz ui bu ui f gvodujpo f jt b *local polynomial-exponential* jg f ? $\prod_{i=1}^n p_i \times m_i$- xi fsf $p_1, \ldots, p_n$ bsf mpdbm qpmzopnjbmt boe $m_1, \ldots, m_n$ bsf fyqpofoujbmt0 Mfu V cf b wbsjfuz po G0 Yf tbz ui bu *local spectral synthesis* i pmet jo V jg ui f tfu pg mpdbm qpmzopnjbm fyqpofoujbmt dpoubjofe jo V jt efotf jo V0 Yf tbz ui bu *local spectral synthesis* i pmet po b hspvq G jg mpdbm tqfdusbm tzoui ftjt i pmet jo fwfsz wbsjfuz po G0 Ui f gpmmpxjoh sftvmu xbt qspwfe jo]29‘0

Theorem 2.22. *There exists a cardinal $\omega_1 \geq \kappa \geq 3^\omega$ such that, for every Abelian group G, local spectral synthesis holds on G if and only if $r_0)G+ < \kappa$. In particular, local spectral synthesis holds on every countable Abelian group G.*

Bt ui f qsppg pg Mfgsbod(t ui fpsfn cbtfe po ui f sfevdujpo up b dpotfrvfodf pg b Lsvmm(t joufstfdujpo ui fpsfn - ui f qsppg pg ui jt tubufn fou jt cbtfe po b hfofsbmj-bujpo pg Lsvmm(t joufstfdujpo ui fpsfn 0

Yf kvtu sfn bsl ui bu ui f fybdu wbmvf pg κ jt vol opxo0 Ui f dpokfduvsf gpsn vmbufe jo]29‘ jt ui bu κ ? ω_1- xi jdi jt frvjwbmfou up ui f tubufn fou ui bu mpdbm tqfdusbm tzoui ftjt epft opu i pme po ui f gsff Bcfmjbo hspvq hfofsbufe cz ω_1 fmfn fout0

Gps ui f tbl f pg dpn qmfufoftt xf rvpuf ui f gpmmpxjoh sftvmu gspn]29‘ xi jdi jt ui f bobmphvf pg ui f dbtf pg atuboebse(tqfdusbm tzoui ftjt0

Proposition 2.23. *If local spectral synthesis holds on the Abelian group G, then the same is true for every homomorphic image of G.*

2.3 Linear functional equations

Mfu a_i, b_i, c_i cf ffiyfe dpn qmfy ovn cfst boe K cf b tvcfffm pg uif fffm pg dpn qmfy ovn cfst xijdi dpoubjot uif qbsbn fufst b_i, c_i0 Jo uijt uiftjt pvs n bjo dpodfso jt uif jowftujhbujpo pg uif tpmvujpot pg uif gvodujpobm frvbujpot pg uif gpsn

$$\prod_{i=1}^{n} a_i f)b_i x\ ,\ \ c_i y+?\ 1 \qquad)x, y\ /\ K+ \qquad)5+$$

xifsf f ; $K \Leftarrow \mathbb{C}$ jt uif vol opxo gvodujpo0

Pvs bjn jt up eftdsjcf uif tpmvujpot- bu mfbtu po uif fffm K. Jo uif gpmmpxjoh tfdujpot xf hjwf b eftdsjqujpo pg bmm tpmvujpot pg frvbujpo)5+tbujtgzjoh dfsubjo obuvsbm dpoejujpot0

Opuf uibu *every solution defined on K can be extended to a solution on* $\mathbb{C}$. Joeffe- tjodf $\mathbb{C}$ jt b mjofbs tqbdf pwfs uif fffm K- uif jefoujuz po K dbo cf fyufoefe up $\mathbb{C}$ bt b mjofbs gvodujpo pwfs K. Mfu ϕ ; $\mathbb{C} \Leftarrow K$ cf tvdi bo fyufotjpo0 Ju jt dmfbs uibu jg f tbujtfffft)5+gps fwfsz $x, y\ /\ K$- uifo $f \equiv \phi$ tbujtfffft)5+gps fwfsz $x, y\ /\ \mathbb{C}$0 Yf opuf bmtp uibu *if there is a non-identically zero solution on* $\mathbb{C}$*, then there is one on* K. Uijt jt pcwjpvt gspn uif gbdu uibu jg f jt b tpmvujpo- uifo tp jt $f)c \times x+$gps fwfsz $c\ /\ \mathbb{C}$0

Gjstu xf tibmm sftusjdu pvs buufoujpo up uiptf frvbujpot)5+xiptf qbsbn fufst tbujtgz uif gpmmpxjoh dpoejujpo0

$$\begin{array}{l}\textit{The numbers } a_1, \dots, a_n \textit{ are nonzero, and } \textbf{there exists } \textit{an } 2 \geq i \geq n \\ \textit{such that } \ b_i c_j \,?\ \ b_j c_i \textit{ holds for any } 2 \geq j \geq n,\ \ j \,?\ \ i.\end{array} \qquad)6+$$

Uif gpmmpxjoh sftvmu xbt sfejtdpwfsfe tfwfsbm ujn ft0 Uifsf bsf n vdi n psf hfofsbm wbsjbout)tff uif Sfn bsl t gpmmpxjoh uif uifpsfn +0 Tjodf uif sftvmu jt pg cbtjd jn qpsubodf jo pvs jowftujhbujpot- xf qspwjef b tjn qmf qsppg0

Theorem 2.24. *Suppose that*

$$\textbf{for every } 2 \geq i \geq n,\ b_i c_j \,?\ \ b_j c_i \textit{ holds for any } \ 2 \geq j \geq n,\ \ j \,?\ \ i. \qquad)7+$$

Let K be a subfield of complex numbers which contains b_i and c_i. If the functions f_i ; $K \Leftarrow \mathbb{C}$ $)i\ ?\ \ 2, \dots, n+$satisfy

$$\prod_{i=1}^{n} f_i)b_i x\ ,\ \ c_i y+?\ 1 \qquad)8+$$

for every $x, y\ /\ K$, then every f_i is a generalized polynomial on K of degree at most n 3.

Proof. Uif qsppg jt cbtfe po uif gpmmpxjoh jefb0 Mfu $d_{i,j}\ ?\ \ b_i c_j \quad b_j c_i$ gps fwfsz $i\,?\ \ j\ /$ $\}2, \dots, n|$ 0 Cz dpoejujpo)7+ $d_{i,j}\,?\ \ 1$0 Uifo

$$b_k)x\ ,\ \ c_n h+,\ \ c_k)y \quad b_n h+?\ b_k x\ ,\ \ c_k y\ ,\ \ d_{k,n} h$$

gps fwfsz x, y, h / K boe k ? $2, \ldots, n$0 Ui fsf gpsf-

$$
\begin{aligned}
1 \; ? \; & \prod_{i=1}^{n} f_i)b_i)x \;,\; c_n h+,\; c_i)y \quad b_n h++ \; \prod_{i=1}^{n} f_i)b_i x \;,\; c_i y+ \\
? \; & \prod_{i=1}^{n})f_i)b_i x \;,\; c_i y \;,\; d_{i,n}h+ \quad f_i)b_i x \;,\; c_i y++ \\
? \; & \prod_{i=1}^{n} \Lambda_{d_{i,n}h} f_i)b_i x \;,\; c_i y+ ? \prod_{i=1}^{n \;\; 1} \Lambda_{d_{i,n}h} f_i)b_i x \;,\; c_i y+
\end{aligned}
$$

Ui f rbtu frvbujpo i pmet- tjodf $d_{n,n}$? 10 Ui vt ui fsf bsf n 2 ufsn t sfn bjofe0 Cz dpoujovjoh ui f qspdftt xf hfu ui f gpmpx joh;

$$
1 \; ? \prod_{i=1}^{n \;\; j \;\; 1} \Lambda_{d_{i,n-j}h_j} \Lambda_{d_{i,n-j+1}h_{j-1}} \ldots \Lambda_{d_{i,n}h_1} f_i)b_i x \;,\; c_i y+,
$$

gps fwfsz j ? $2, \ldots, n$ 20 Bu ui f foe pg ui jt qspdftt)j ? n 2+xf pcubjo

$$
1 \; ? \; \Lambda_{d_{1,2}h_{n-1}} \Lambda_{d_{1,3}h_{n-2}} \ldots \Lambda_{d_{1,n}h_1} f_1)b_1 x \;,\; c_1 y+
$$

Tjodf $d_{1,2}, \ldots, d_{1,n}$ bsf opo–fsp boe ui f ovn cfst h_i)i ? $2, \ldots, n$ 2+dbo cf di ptfo bscjusbsjmz- ui jt ti px t ui bu f_1 jt b hfofsbmj{fe qpmzopn jbmpg efhsff bu n ptu n 30 Cz tzn n fusz- xf pcubjo ui bu f_i jt b hfofsbmj{fe qpmzopn jbmpg efhsff bu n ptu n 3 gps fwfsz i ? $2, \ldots, n$0 □

Remark 2.25. Ui f qsfwjpvt ui fpsfn i bt tfwfsbmhfofsbmj{bujpot0 Pof pg ui fn jt ui f sftvmu pg M0 T–fl f mzi jej]44- Ui fpsfn 40 ')tff bmtp]2'-]28' boe]47'+0

Theorem 2.26. *Let G, S be Abelian groups, and suppose that G is divisible and S is torsion-free. Let n be a non-negative integer, and let ϕ_i, ψ_i ; $G \Leftarrow G$ be homomorphisms of G onto itself such that* S h)$\psi_j \equiv \psi_i^{\;1} \quad \phi_j \equiv \phi_i^{\;1}$+? G *for* i ? j)i, j ? $2, \ldots, n$+, *where* S h)ϕ+*denotes the range of* ϕ. *If the functions* f_i ; $G \Leftarrow S$)i ? $1, \ldots, n$+*satisfy*

$$
f_0)x+, \; \prod_{i=1}^{n} f_i)\phi_i)x+, \;\; \psi_i)y++ ? \; 1,
$$

then each f_i is a generalized polynomial of degree at most n 2.

Bt b tqfdjbmdbtf pg ui jt ui fpsfn xf hfu ui f gpmpx joh0 *Let V and W be vector spaces over one of the fields $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, and let a_i, b_i, c_i)i ? $2, \ldots, n$+be scalars which satisfy*)7+. *If the functions f_i ; $V \Leftarrow W$)i ? $2, \ldots, n$+satisfy equation*)8+*for every x, y / V, then every f_i is a generalized polynomial of degree at most n 3.*

Jo pvs jowftujhbujpot xf ti bmoffe ui f gpmpx joh wfstjpo pg Ui fpsfn 30850 Ui f bshvn fou vtfe jo ui f qsppg pg Ui fpsfn 3085 jo ui f dbtf xi fo dpoejujpo)7+ jt sfqmbdfe cz $d_{i,j}$? 1)j ? i+ hjwft ui bu f_i jt b hfofsbmj{fe qpmzopn jbm pg efhsff $\geq n$ 30 Jg f_1 ? ... ? f_n ui fo xf pcubjo ui f gpmpx joh0

Theorem 2.27. *Let n be a positive integer, and b_i, c_i)i ? $2, \ldots, n$+ complex numbers which satisfy condition)6+. Let K be a subfield of the complex numbers containing b_i, c_i, and let f ; $K \Leftarrow \mathbb{C}$ be a function satisfying*

$$\prod_{i=1}^{n} a_i f)b_i x \,, \quad c_i y+? \;\; 1 \qquad)9+$$

for every x, y / K. Then f is a generalized polynomial on K of degree at most n 3.

Dpoejujpo)6+ jt bvupn bujdbmmz tbujtfjfe jo tpn f jn qpsubou tqfdjbm dbtft- tvdi bt

$$\prod_{i=1}^{n} a_i f)b_i x \,, \quad y+? \;\; 1 \qquad):+$$

boe

$$\prod_{i=1}^{n} a_i f)b_i x \,, \quad)2 \quad b_i+ y+? \;\; 1, \qquad)21+$$

xi fsf $b_1, \ldots, b_n$ bsf ejtujodu0

Ui f gpmpx joh fybn qmf ti px t ui bu dpoejujpo)6+ jt ofdfttbsz jo Ui fpsfn 30880

Proposition 2.28. *The functional equation*

$$f)x+ \quad f)3x+? \;\; 1 \qquad)x \;/\; \mathbb{C}+ \qquad)22+$$

has nonconstant solutions, although every generalized polynomial solution of this equation is constant.

Proof. Mfu H_x efopuf ui f tfu }$3^n x$; n / $\mathbb{Z}$| gps fwfsz x / $\mathbb{C}$. Ju jt dmfbs ui bu gps x ? x^∞ fjui fs H_x ? $H_{x'}$ ps H_x { $H_{x'}$? $\mathcal{A}$ Cz [pso(t mfn n b- ui fsf fyjtut b n byjn bm tvctfu X pg $\mathbb{C}$ tvdi ui bu $\sum_{x \emptyset X} H_x$? $\mathbb{C}$ boe H_x { $H_{x'}$? $\mathcal{A}$ gps fwfsz x, x^∞/ X0 Jg xf efgjof f po fwfsz H_x)x / X+ bt bo bscjusbsz dpotubou- ui fo xf pcubjo b tpmvujpo pg)22+0

Po ui f pui fs i boe xf ti px ui bu fwfsz hfofsbmj{fe qpmzopn jbm tpmvujpo pg $f)x+ \;\; f)3x+?$ 1 jt dpotubou0 Mfu vt bttvn f ui bu g jt b opodpotubou hfofsbmj{fe qpmzopn jbm tpmvujpo0 Ui fo g jt pg ui f gpsn $\prod_{j=0}^{m} f_j$- xi fsf f_j jt b n popn jbm pg efhsff j gps j ? $1, \ldots, m$0 Ui jt n fbot ui bu f_j jt b ejbhpobm pg b j.beejujwf gvodujpo0 Tvqqptf ui bu $m \subset 2$ boe f_m jt opu jefoujdbmmz {fsp0 Tjodf boz j.beejujwf gvodujpo i bt ui f sbujpobm i pn phfofjuz qspqfsuz jo fwfsz dppsejobuf- xf i bwf

$$g)rx+? \;\;)f_0 \,, \;\; r \times f_1)x+, \;\; \ldots, \;\; r^m \times f_m)x++? \;\;)f_0 \,, \;\; 3r \times f_1)x+, \;\; \ldots, \;\; 3^m r^m \times f_m)x++? \;\; g)3rx+$$

gps fwfsz x / $\mathbb{C}$, r / $\mathbb{Q}$. Mfu x / $\mathbb{C}$ cf ffiyfe tvdi ui bu $f_m)x+?$ 10 Ui fo xf hfu

$$r \times f_1)x+, \quad \ldots, \quad)3^m \quad 2+r^m \times f_m)x+? \; 1.$$

Ui jt jt b opo–fsp qpmzopn jbmpg ui f wbsjbcmf r- x i jdi i bt joffiojufm n boz sppu t0 Ui jt dpousbejdujpo ti px t ui bu g jt dpotubou0 □

Y f opuf ui bu ui f tubufn fou pg Qspqptjujpo 3039 dbo cf hfofsbmj-fe bt gpmmpx t0 *Suppose that the pairs of parameters (b_i, c_i) lie on a line going through the origin. Then the functional equation* $)9+$ *can be reduced to one of the form* $\prod_{i=1}^n a_i f)d_i x+? \; 1$, *which only has constant generalized polynomial solutions.*

Ui f tqbdf pg ui f tpmvujpot pg $)9+$ efffiof e po $\mathbb{C}$ jt bmx bzt b mjofbs tqbdf pwfs $\mathbb{C}$0 Ju jt f btz up tff ui bu *the space of solutions of any of the equations* $): +,)21+$ *is translation invariant.* I px f wfs- jo hfofsbmui f tqbdf pg ui f tpmvujpot pg $)9+$ jt opu ofdfttbsz usbotmbujpo jowbsjbou bt ti px o cz ui f frvbujpo pg Qspqptjujpo 30390

Jo hfofsbm- *if the points* $)b_i, c_i+ / \mathbb{C}^2$ *lie on a line not going through the origin* $)1, 1+$, *then the space of solutions is translation invariant.*

Joeffe- ju gpmmpx t gspn ui f dpoejujpo ui bu ui fsf bsf dpotubout β, γ / $\mathbb{C}$ tvdi ui bu $\beta b_i \; , \; \gamma c_i \; ? \; 2 \;)i \; ? \; 2, \ldots, n+0$ Jg f; $\mathbb{C} \Leftarrow \mathbb{C}$ jt b tpmvujpo pg $)9+$ ui fo x f i bwf

$$\prod_{i=1}^n a_i T_h f)b_i x \; , \; c_i y+? \; \prod_{i=1}^n a_i f)b_i x \; , \; c_i y \; , \; h+?$$
$$? \; \prod_{i=1}^n a_i f)b_i x \; , \; c_i y \; , \; \beta b_i h \; , \; \gamma c_i h+?$$
$$? \; \prod_{i=1}^n a_i f)b_i)x \; , \; \beta h+, \; c_i)y \; , \; \gamma h++? \; 1$$

gps fwfsz h, x, y / $\mathbb{C}$0 Ui vt $T_h f$ jt bmtp b tpmvujpo- x i jdi qspwft usbotmbujpo jowbsjbodf0

Y f opuf ui bu boz frvbujpo gps x i jdi ui f qpjout $)b_i, c_i+ / \mathbb{C}^2$ mjf po b mjof opu hpjoh ui spvhi ui f psjhjo $)1, 1+$ dbo cf usbotgpsn fe joup bo frvbujpo pg uzqf $): +$ vtjoh b tvjubcmf mjofbs tvctujuvujpo0 Gps jotubodf- ui f tvctujuvujpo $x \Leftarrow x \; , \; y$ usbotgpsn t $)21+$ joup $): +0$ Ui f usbotmbujpo jowbsjbodf pg ui f tqbdf pg tpmvujpot pg ui f frvbujpot tbujtgzjoh ui jt dpoejujpo gpmmpx t gspn ui jt pctfswbujpo0

I px f wfs- ui f dpmmjofbsjuz pg $)b_i, c_i+$ jt opu ofdfttbsz gps ui f tvctqbdf pg tpmvujpot up cf usbotmbujpo jowbsjbou0 Ui jt jt ti px o cz ui f gpmmpx joh qspqptjujpo0

Proposition 2.29. *There is a functional equation of the type* $)5+$ *which only has constant solutions, and the pairs* $)b_i, c_i+$ *are not collinear.*

Ui f qsppg pg ui jt tubufn fou vtft tpn f sftvmt pg ui f ofyu tfdujpo- ui vt xf qptuqpof ui f qsppg voujmQspqptjujpo 40340

From now on we concentrate on the generalized polynomial solutions of the equations)9+.)Cz Ui fpsfn 3038- jg dpoejujpo)6+jt tbujtfffe ui fo bmmtpmvujpo bsf bvupn bujdbmmz hfofsbmj-fe qpmzopn jbmt0+

Lemma 2.30)]26'+. *Let f ? f_0 , $\prod_{k=1}^{m} f_k$, where f_0 is constant and f_k is a monomial of degree k)k ? $2, \ldots, m$+. Then f is a solution of)9+ if and only if each of $f_0, \ldots, f_m$ is a solution of)9+.*

Proof. Jgfbdi pg $f_0, \ldots, f_m$ jt b tpmvujpo ui fo tp jt f bt ui f tqbdf pg tpmvujpot jt mjofbs0

Tvqqptf f jt b tpmvujpo0 Tjodf boz k.beejujwf gvodujpo i bt ui f sbujpobmi pn phfofjuz qspqfsuz jo fwfsz dppsejobuf- xf i bwf

$$\prod_{i=1}^{n} a_i \times) f_0 , \quad r \times f_1)b_i x , \quad c_i y+, \quad \ldots , \quad r^m \times f_m)b_i x , \quad c_i y++ ? \ 1$$

gps fwfsz x, y / $\mathbb{C}$, r / $\mathbb{Q}$. Mfu x, y cf ffiyfe0 Qvuujoh F_j ? $\prod_{i=1}^{n} a_i f_j)b_i x$, $c_i y$+xf pcubjo $\prod_{j=0}^{m} F_j \times r^j$? 1 gps fwfsz r / $\mathbb{Q}$0 Ui vt F_j ? 1 gps fwfsz j- qspwjoh ui bu f_j jt b tpmvujpo pg)9+0 $\square$

Ui f dpotubou tpmvujpot pg)9+bsf usjwjbmmp ffioe; jg $\prod_{i=1}^{n} a_i$? 1 ui fo fwfsz dpotubou gvodujpo jt b tpmvujpo=pui fsxjtf ui fsf jt op opo–fsp dpotubou tpmvujpo0

Ju jt bmtp dmfbs ui bu jg f jt beejujwf- ui fo f jt b tpmvujpo pg)9+jg boe pomz jg

$$\prod_{i=1}^{n} a_i \times f)b_i x+ ? \prod_{i=1}^{n} a_i \times f)c_i x+ ? \ 1$$

gps fwfsz x0 Opuf ui bu jo ui f dbtf pg): +boe)21+ $\prod_{i=1}^{n} a_i$? 1 jt b ofdfttbsz dpoejujpo pg ui f fyjtufodf pg opo–fsp beejujwf tpmvujpot0

Ui f k.beejujwf tpmvujpot pg n psf hfofsbmfrvbujpot i bwf cffo jowftujhbufe cz T–flfmz. i jej]47' boe Wbshb boe Wjod–f]53'0

Jo]49' ui f frvbujpo

$$\prod_{i=1}^{n} a_i \times f)b_i x+ ? \ 1 \qquad)23+$$

jt dbmmfe Ebspd–z(t gvodujpobmfrvbujpo0 Ui jt ufsn jopmphz dpn ft gspn ui f gpmmpx joh ui fp. sfn pg [0Ebspd–z]8'0

Theorem 2.31. *The equation $a \times f)x+$ $f)bx+ ?$ 1 has a nonzero additive solution if and only if either a and b are transcendental or they are algebraically conjugates.*

)Ebsŏd-z(t ui fpsfn jn qmjft ui buui f frvbujpo $f)x+ \;\; f)3x+? \;\; 1$ i bt op opo–fsp beejujwf tpmvujpo- bt xf i bwf tffo bmsfbez jo Qspqptjujpo 30390+

Y f opuf ui bu ui f gvodujpobmfr vbujpot)21+xfsf jowftujhbufe cz Besjfoo Wbshb]51‘ jo ui f dbtf xi fo a_i, b_i / $\mathbb{R}$ boe)21+jt tbujtfffe gps x, y / I xi fsf $I \leq \mathbb{R}$ jt bo joufswbm0 Ti f vtfe ui f fyufotjpo ui fpsfn pg Qbmft]35‘ boe ti pxfe ui bu jg f jt b tpmvujpo po I ui fo ui fsf fyjtut b vojrvf fyufotjpo pg f up $\mathbb{R}$ xi jdi tbujtfffft)21+gps fwfsz x, y / $\mathbb{R}$0

Y f dmptf ui jt tfdujpo xjui b ojdf sftvmupg B0Wbshb boe Dt0 Wjod-f]52‘ bcpvu Ebsŏd-z(t gvodujpobmfr vbujpo dpoubjojoh tfwfsbmufsn t0

Theorem 2.32. *Suppose that* $n \subset 3$.

)j+ *Let us assume that the parameters* $b_1, \ldots, b_n$ *are algebraically independent over* $\mathbb{Q}$. *Then the equation*

$$f)x+, \prod_{i=1}^{n\ \ 1} a_i f)b_i x+? \;\; 1 \tag{)24+}$$

has nonzero additive solutions if and only if at least one of the parameters $a_1, \ldots, a_n$ *is a transcendental number.*

)jj+ *Let us assume that the parameters* $a_1, \ldots, a_n$ *are algebraically independent over* $\mathbb{Q}$. *Then the equation*)24+ *has nonzero additive solutions if and only if at least one of the parameters* $b_1, \ldots, b_n$ *is transcendental number.*

Bt xf ti bmtff jo ui f ofyu tfdujpo- Ui fpsfn t 3042 boe 3043 bsf fbtz dpotfrvfodft pg Ui fpsfn t 408 boe 4025)tff Sfn bsl 40 +0 Gps b gvsui fs hfofsbmj{bujpo- tff Ui fpsfn 60250

3 Existence of nonzero solutions of linear functional equations

3.1 Some varieties related to linear functional equations

Jo ui jt tvctfdujpo xf joupevdf tpn f opubujpo boe cbtjd sftvmut ui bu xjmmcf offefe jo ui f tfrvfmY f dpotjefs ui f mjofbs gvodujpobmfrvbujpo

$$\prod_{i=1}^{n} a_i f)b_i x\,,\ \ c_i y+?\ 1. \qquad)25+$$

Y f ffly b ffbjufm hfofsbufe tvcfffm K pg $\mathbb{C}$ dpoubjojoh ui f qbsbn fufst b_i, c_i)i ? $2, \ldots, n$+0 Y f efopuf cz S ui f tfu pg tpmvujpot pg)25+efffofe po K0 Ju jt dmfbs ui bu S jt b dmptfe mjofbs tvctqbdf pg ui f qspevdu tqbdf $\mathbb{C}^K$0

Mfu S_k efopuf ui f tfu pg ui ptf tpmvujpot pg)25+efffofe po K xi jdi bsf hfofsbufe n popn jbmt pg efhsff k0 Ui vt S_1 jt ui f tfu pg beejujwf tpmvujpot po K0

Y f efopuf cz V_k ui f tfu pg k.beejujwf gvodujpot efffofe po ui f beejujwf hspvq K^k0 Ui vt V_1 jt ui f tfu pg beejujwf gvodujpot po K0

Lemma 3.1. *V_k is a closed linear subspace of the product space $\mathbb{C}^{K^k}$, and S_k is a closed linear subspace of the product space $\mathbb{C}^K$.*

Proof. Ju jt dmfbs ui bu V_k jt b mjofbs tqbdf0 Y f qspwf ui bu V_1 jt dmptfe=b tjn jmbs bshvn fou xpsl t gps V_k0

Mfu $g; K \Leftarrow \mathbb{C}$ cf b gvodujpo jo ui f dmptvsf pg $V_1$0 Tjodf g / clV_1- gps fwfsz $\varepsilon > 1$ ui fsf fyjtut f / V_1 tvdi ui bu

$$\backslash f)a+\ \ g)a+\backslash < \varepsilon,\ \backslash f)b+\ \ g)b+\backslash < \varepsilon \text{ boe } \backslash f)a\,,\ \ b+\ \ g)a\,,\ \ b+\backslash < \varepsilon.$$

Opx- $f)a+,\ \ f)b+?\ \ f)a\,,\ \ b+$cfdbvtf f jt beejujwf- tp $\backslash g)a\,,\ \ b+\ \ g)a+\ \ g)b+\backslash < 4\varepsilon$0 Ui jt jt usvf gps fwfsz ε- ui fsfgpsf $g)a\,,\ \ b+?\ \ g)a+,\ \ g)b+$0 Ui jt i pmet gps fwfsz a, b / K xjui $a, b, a\,,\ \ b$? 1 boe ui fo- cz $g)1+?\ 1$- ju gpmmpx t ui bu g jt beejujwf po K0 Ui jt n fbot ui bu V_1 jt dmptfe0

Mfu ejbh V_k ? }ejbh F ; F / $V_k|$0 Ui fo ejbh V_k jt b dmptfe mjofbs tvctqbdf pg $\mathbb{C}^K$; ui jt gpmmpx t gspn ui f gpsn vmb)3+boe ui f gbdu ui bu V_k jt dmptfe0 Tjodf S_k ? S { ejbh V_k- xf pcubjo ui bu S_k jt dmptfe0 □

Mfu M_k efopuf ui f tfu pg ui f gvodujpot $F; K^k \Leftarrow \mathbb{C}$ tvdi ui bu F jt k.beejujwf- boe ui f gvodujpo $x \not\Leftarrow F)s_1x, s_2x, \ldots, s_kx+$jt b tpmvujpo pg)25+po K gps fwfsz $s_1, s_2, \ldots s_k$ / K }1|0

Yf qvu $K^{\subseteq}$? $\}x \ / \ K \ ; \ x$ î? $1|$ =uifo $K^{\subseteq}$ jt bo Bcfmjbo hspvq voefs nvmujqmjdbujpo0 Uif Bcfmjbo hspvq $\underbrace{K^{\subseteq} * \ldots * K^{\subseteq}}_{k}$ xjmm cf efopufe cz $)K^{\subseteq}+^k$- xifsf uif pqfsbujpo jt nvmujqmjdbujpo jo fwfsz dppsejobuf pg uif wfdupst pg $)K^{\subseteq}+^k$0 Yf qvu $V_k^{\subseteq}$? $\}F\backslash_{(K^*)^k} \ ; \ F \ / \ V_k|$ boe

$$M_k^{\subseteq} ? \ \}F\backslash_{(K^*)^k} \ ; \ F \ / \ M_k| .$$

Lemma 3.2. *$V_k^{\subseteq}$ and $M_k^{\subseteq}$ are varieties on $)K^{\subseteq}+^k$.*

Proof. Ju jt dmfbs uibu V_k boe $V_k^{\subseteq}$ bsf mjofbs tqbdft pwfs $\mathbb{C}$0 Usbotmbujpo jowbsjbodf pg $V_k^{\subseteq}$ po $)K^{\subseteq}+^k$ gpmmpxt gspn uif gbdu uibu jg $F \ ; \ K^k \Leftarrow \mathbb{C}$ jt k.beejujwf- uifo tp jt

$$)x_1,\ldots,x_k+ \not\Leftarrow F)c_1x_1,\ldots,c_kx_k+ \qquad))x_1,\ldots,x_k+ / K^k+$$

gps fwfsz $)c_1,\ldots,c_k+ /)K^{\subseteq}+^k$0 Uif tubufnfou uibu V_k boe $V_k^{\subseteq}$ bsf dmptfe dbo cf qspwfe bt jo Mfnnb 402 boe jt mfgu up uif sfbefs0

Ju jt fbtz up tff uibu M_k jt b dmptfe mjofbs tvctqbdf pg V_k- boe $M_k^{\subseteq}$ jt b dmptfe mjofbs tvctqbdf pg $V_k^{\subseteq}$0 Usbotmbujpo jowbsjbodf pg $M_k^{\subseteq}$ po $)K^{\subseteq}+^k$ nfbot uibu jg $F \ / \ M_k^{\subseteq}$- uifo uif nbq $)x_1,\ldots,x_k+ \not\Leftarrow F)c_1x_1,\ldots,c_kx_k+)x_1,\ldots,x_k \ / \ K^{\subseteq}+$bmtp cfmpoht up $M_k^{\subseteq}$ gps fwfsz $c_1,\ldots,c_K \ / \ K^{\subseteq}$- xijdi jt fbtjmz tffo gspn uif efgojujpo pg $M_k^{\subseteq}$0 □

Lemma 3.3.

$$S_k \ ? \ \}\mathrm{ejbh}\, F \ ; \ F \ / \ M_k| .$$

Proof. Ju jt dmfbs uibu ejbh $F \ / \ S_k$ gps fwfsz $F \ / \ M_k$0 Yf qspwf uif dpowfstf0 Mfu $f \ ; \ K \Leftarrow \mathbb{C}$ cf bo fmfnfou pg S_k0 Uifo f jt b tpmvujpo)25+po K- boe f ? ejbh F gps b tznnfusjd k.beejujwf gvodujpo $F \ ; \ K^k \Leftarrow \mathbb{C}$0 Yf qspwf $F \ / \ M_k$0 Yf ibwf up tipx uibu gps fwfsz $)s_1,\ldots,s_k+ /)K^{\subseteq}+^k$ uif ejbhpobm pg uif gvodujpo

$$G)x_1,\ldots,x_k+? \ F)s_1x_1,\ldots,s_kx_k+$$

cfmpoht up S_k0 Mfu ejbh G ? g0 Uifo- cz Mfnnb 302 xf ibwf

$$g)x+? \ F)s_1x,\ldots,s_kx+? \ \frac{2}{k"} \times \Lambda_{s_1x}\Lambda_{s_2x}\ldots\Lambda_{s_kx}f)1+? \ \prod_{j=1}^{M} \circ f)e_jx+$$

xjui tvjubcmf $e_1,\ldots,e_M \ / \ K$. Tjodf $x \not\Leftarrow f)ex+$cfmpoht up S_k gps fwfsz $e \ / \ K$- ju gpmmpxt uibu $g \ / \ S_k$- boe uivt $F \ / \ M_k$0 □

Uif gpmmpxjoh qspqptjujpo xjmm cf vtfe gsfrvfoumz)tff]27- Uifpsfn 250602- q0 469‘+0

Proposition 3.4. *Let $K \to \mathbb{C}$ be a finitely generated field and $\phi; K \Leftarrow \mathbb{C}$ be an injective homomorphism. Then there exists an automorphism ψ of $\mathbb{C}$ such that $\psi\backslash_K \ ? \ \phi$.*

Lemma 3.5. *If* m / $V_1^{\subseteq}$ *is an exponential on* $K^{\subseteq}$*, then* m *can be extended to* $\mathbb{C}$ *as an automorphism of* $\mathbb{C}$.

Proof. Ui f dpoejujpo m / $V_1^{\subseteq}$ n fbot ui bu fyufoejoh m up K cz $m)1+?$ 1- xf pcubjo bo beejujwf gvodujpo0 Opx m jt bo fyqpofoujbmpo $K^{\subseteq}$- boe ui vt m tbujtfff t $m)xy+?$ $m)x+m)y+$ gps fwfsz x, y / $K^{\subseteq}$0 Dpotfr vfoum- ui f fyufoefe m jt b i pn pn psqi jtn pg K0 Cz m i? 1 po $K^{\subseteq}$- ju gpmpx t ui bu m jt jokfdujwf0 Tjodf ui f usbotdfoefodf efhsff pg K pwfs $\mathbb{Q}$ jt ffojuf-ju gpmpx t- cz Qspqptjujpo 406- ui bu m dbo cf fyufoefe up $\mathbb{C}$ bt bo bvupn psqi jtn pg $\mathbb{C}$0 □

Lemma 3.6. *Suppose that* m / V_k *and* $m\backslash_{(K^*)^k}$ *is an exponential; i.e.* m *is nonzero on* $)K^{\subseteq}+^k$*, and* $m)xy+?$ $m)x+m)y+$ *for every* x, y / $)K^{\subseteq}+^k$*. Then there are field automorphisms* $m_1, \ldots, m_k$ *of* $\mathbb{C}$ *such that*

$$m)x+? \;\; m)x_1, \ldots, x_k+? \;\; m_1)x_1+\times\times\times m_k)x_k+ \qquad)x_1, \ldots, x_k \;/\; K+\!.$$

Proof. Cz ui f n vmujqmjdbujwjuz pg m-

$$m)x_1, \ldots, x_k+? \;\; m)x_1, 2, \ldots, 2+\times m)2, x_2, 2, \ldots, 2+\times \ldots \times m)2, \ldots, 2, x_k+\!.$$

Tjodf x_i $\not\Leftarrow$ $m)2, \ldots, 2, x_i, 2, \ldots, 2+$ jt beejujwf po K boe fyqpofoujbmpo $K^{\subseteq}$- ju jt bo jokfdujwf i pn pn psqi jtn - xi jdi xf efopuf cz m_i0 Cz Qspqptjujpo 406- $m_1, \ldots, m_k$ dbo cf fyufoefe up $\mathbb{C}$ bt bvupn psqi jtn t pg $\mathbb{C}$0 □

3.2 Non-zero additive solutions

Pvs n bjo sftvmu jo ui jt tvctfdujpo jt Ui fpsfn 4080

Mfu K cf b ffojufmz hfofsbufe tvcffme pg $\mathbb{C}$ dpoubjojoh b_i, c_i $)i ?$ $2, \ldots, n+0$ Sfdbmmui bu S_1 efopuft ui f tfu pg beejujwf tpmvujpot pg)25+efffofe po K0 Dmfbsm- f ; $K \Leftarrow \mathbb{C}$ cfmpoht up S_1 jg boe pom jg

$$\prod_{i=1}^{n} a_i f)b_i x+? \;\; 1, \qquad \prod_{i=1}^{n} a_i f)c_i x+? \;\; 1 \qquad)26+$$

i pmet gps fwfsz x / K0

Ju jt fbtz up tff ui bu jg ϕ jt bo bvupn psqi jtn pg $\mathbb{C}$ ui fo ϕ jt b tpmvujpo pg)25+jg boe pom jg

$$\prod_{i=1}^{n} a_i \phi)b_i+? \;\; 1 \text{ boe } \prod_{i=1}^{n} a_i \phi)c_i+? \;\; 1. \qquad)27+$$

Joeffe- xf i bwf- gps fwfsz x, h / $\mathbb{C}$-

$$\prod_{i=1}^{n} a_i \phi)b_i x \,, \;\; c_i h+? \;\; \prod_{i=1}^{n} a_i \,)\phi)b_i x+, \;\; \phi)c_i h++?$$

$$? \Bigg)\prod_{i=1}^{n} a_i\phi)b_i+\Big(\phi)x+, \Bigg)\prod_{i=1}^{n} a_i\phi)c_i+\Big(\phi)h+$$

Theorem 3.7. *There is a nonzero additive solution of)25+ if and only if there exists a solution of)25+ which is an automorphism $\phi; \mathbb{C} \Leftarrow \mathbb{C}$ or, equivalently, an automorphism satisfying)27+.*

Proof. Ui f (jg(tubufn fou jt pcwjpvt0

Mfu f_1 cf b opo–fsp beejujwf tpmvujpo pg)25+ boe mfu d / $\mathbb{C}$ cf tvdi ui bu $f_1)d+$? 10 Qvu K ? $\mathbb{Q})b_1, \ldots, b_n, c_1, \ldots, c_n, d+$ ui f fyufotjpo pg $\mathbb{Q}$ cz ui f dpn qmfy ovn cfst b_i, c_i, d $)i$? $2, \ldots, n+0$

Opuf ui bu S_1 ? M_1- boe ui vt $S_1^{\subseteq}$? $M_1^{\subseteq}0$ Ui fsfgpsf- cz Mfn n b 403- $S_1^{\subseteq}$ jt b wbsjfuz po ui f Bcfmjbo hspvq $K^{\subseteq}0$

Yf i bwf $S_1^{\subseteq}$? }1| cfdbvtf $f_1\backslash_{K^*}$ / $S_1^{\subseteq}$ boe f_1 jt opu jefoujdbmmz –fsp po $K^{\subseteq}0$ Cz Ui fpsfn 3029- jg G jt b ejtdsfuf Bcfmjbo hspvq pg upstjpo gsff sbol mftt ui bo dpoujovvn - ui fo i bsn pojd bobmztjt i pmet po $G0$ Ui jt n fbot ui bu fwfsz opo–fsp wbsjfuz po $\mathbb{C}^G$ dpoubjot bo fyqpofoujbm0 Tjodf $K^{\subseteq}$ jt dpvoubcmf- xf ffoe ui bu $S_1^{\subseteq}$ dpoubjot bo fyqpofoujbm0

Mfu ϕ cf bo fyqpofoujbmfmfn fou pg $S_1^{\subseteq}0$ Tjodf ϕ / $S_1^{\subseteq}$- xf i bwf

$$1 ? \prod_{i=1}^{n} a_i\phi)b_ix+? \prod_{i=1}^{n} a_i\phi)b_i+\phi)x+ \text{boe } 1 ? \prod_{i=1}^{n} a_i\phi)c_ix+? \prod_{i=1}^{n} a_i\phi)c_i+\phi)x+$$

gps fwfsz x / $K^{\subseteq}0$ Ui jt jn qmjft 1 ? $\prod_{i=1}^{n} a_i\phi)b_i+?$ $\prod_{i=1}^{n} a_i\phi)c_i+0$ Vtjoh Mfn n b 406- ϕ dbo cf fyufoefe up bo bvupn psqi jtn pg $\mathbb{C}0$ Ui jt dpn qmfuft ui f qsppg0 □

Remark 3.8. Fttfoujbmmz ui f tbn f qsppg ti px t ui f gpmmpx joh n psf hfofsbmsftvmu0

The functional equation

$$\prod_{i=1}^{n} a_if)b_{i,1}x_1 \;,\; b_{i,2}x_2 \;,\; \times\times\times,\; b_{i,k}x_k+? \;1 \qquad)x_1, \ldots, x_k \;/\; \mathbb{C}+ \qquad)28+$$

has a nonzero additive solution if and only if there exists an automorphism ϕ of $\mathbb{C}$ which is a solution.

Jo gbdu- *all of our results to be presented about the generalized polynomial solutions of)25+ can be generalized, with the same proofs, to the equations of the form)28+ for any k.*

Remark 3.9. Ui fpsfn 3042 jt bo jn n fejbuf dpotfrvfodf pg Sfn bsl 4090 Joeffe- jg ui fsf jt b opo–fsp beejujwf tpmvujpo pg frvbujpo $a \times f)x+ \;\; f)bx+?$ 1- ui fo ui fsf fyjtut bo bvupn psqi jtn ϕ pg $\mathbb{C}$ xi jdi jt b tpmvujpo pg ui jt frvbujpo0 Ui vt- ui f frvbujpo $a \times\phi)x+$ $\phi)bx+?$ 1 jn qmjft ui bu a ? $\phi)b+$ xi jdi n fbot ui bu fjui fs a boe b bsf usbotdfoefoubmps ui fz bsf bmhfcsbjdbmmz dpokvhbuft0

Theorem 3.10. *Suppose that $a_1,\dots,a_n$ are nonzero and $b_1,\dots,b_n$ are distinct complex numbers. The following are equivalent:*

)j+ *There is a nonconstant solution of*

$$\prod_{i=1}^{n} a_i f)b_i x\,,\ \ y+?\ 1\quad x,y\ /\ \mathbb{C}. \qquad)29+$$

)jj+ *There is a solution of*)29+*which is an automorphism of* $\mathbb{C}$.

)jjj+ $\prod_{i=1}^{n} a_i\ ?\ 1$, *and there exists an automorphism* $\phi;\ \mathbb{C}\Leftarrow\mathbb{C}$ *such that*

$$\prod_{i=1}^{n} a_i \phi)b_i+?\ 1.$$

Proof.)j+?⇔)jj+; Ui f tvctujuvujpo $x\ ?\ 1$ ti pxt ui bu $\prod_{i=1}^{n} a_i f)y+?\ 1$, ui vt fjui fs $\prod_{i=1}^{n} a_i\ ?\ 1$ ps ui f 1 gvodujpo jt ui f pomz tpmvujpo0 Cz)j+ xf i bwf $\prod_{i=1}^{n} a_i\ ?\ 1$0 Ui jt dpoejujpo jn qmjft ui bu ui f dpotubou gvodujpot bsf tpmvujpot0

Mfu $f)x+$cf b opodpotubou tpmvujpo0 Cz Ui fpsfn 3035- fwfsz tpmvujpo jt b hfofsbmj–fe qpmzopn jbm0 Jg ui f efhsff pg f jt k- ui fo gps tvjubcmf $h_1,\dots,h_k$ 1- ui f efhsff pg ui f gvodujpo $g\ ?\ \Lambda_{h_1}\dots\Lambda_{h_{k-1}} f$ jt 20 Ui fo $g)x+?\ a)x+,\ c$- xi fsf $a)x+$jt b opo–fsp beejujwf gvodujpo boe c jt b dpotubou0 Tjodf $g\quad c$ jt b tpmvujpo- xf n bz bttvn f ui bu f jt b opu jefoujdbmmz –fsp beejujwf gvodujpo0 Ui fo- cz Ui fpsfn 408- ui fsf jt bo bvupn psqi jtn pg $\mathbb{C}$ xi jdi jt b tpmvujpo pg)63+0

)jj+?⇔)jjj+; Jg ϕ jt bo bvupn psqi jtn xi jdi jt b tpmvujpo pg)63+ ui fo $\prod_{i=1}^{n} a_i\phi)2+?\ 1$ boe $\prod_{i=1}^{n} a_i\phi)b_i+?\ 1$.

)jjj+?⇔)j+; Ju jt dmfbs cz ui f qsfwjpvt ui fpsfn 0 □

Remark 3.11. Ui fpsfn 4021 epft opu i pme jo ui f hfofsbmdbtf- cfdbvtf ui f fyjtufodf pg b opo–fsp tpmvujpo pg)25+epft opu jn qmz ui f fyjtufodf pg b opo–fsp beejujwf tpmvujpo0 Gps fybn qmf- ui f frvbujpo

$$f)x\,,\ y+,\ f)x\quad y+\quad 3f)x+\quad 3f)y+?\ 1$$

jt tbujtgjfe cz x^2. Ju jt fbtz up di fdl ui bu fwfsz beejujwf tpmvujpo n vtu cf –fsp0

Remark 3.12. Yf ti bmmoffe ui f gpmmpxjoh- tmjhi umz tuspohfs wfstjpo pg ui f jn qmjdbujpo)j+?⇔)jj+pg Ui fpsfn 40210 *Suppose that there is a nonconstant function $f;\ K\Leftarrow\mathbb{C}$ such that*)29+*holds for every $x\ /\ K^{\subsetneq}$, $y\ /\ K$. Then there is an automorphism solution to*)29+; Ui jt dbo cf qspwfe cz vtjoh ui f bshvn fou pg Ui fpsfn t 408 boe 4021- tjodf ui f tfu pg gvodujpot tbujtgzjoh ui f dpoejujpo bcpwf bmtp gpsn t b wbsjfuz po $K^{\subsetneq}$0

Proposition 3.13. *Suppose* $\prod_{i=1}^{n} a_i$? 1 *and* $\prod_{i=1}^{n} a_ib_i$ ≠? 1. *If either*

)j+ $b_i \,/\, \mathbb{Q}$)i ? 2, . . . , n+, *or*

)jj+ $a_i \,/\, \mathbb{Q}$)i ? 2, . . . , n+,

then every solution of $\prod_{i=1}^{n} a_i f)b_ix$, y+? 1 *is constant.*

Proof. Yf n bz bttvn f ui bu $a_1, \ldots, a_n$ bsf opo–fsp boe $b_1, \ldots, b_n$ bsf ejtujodu0 Joeffe- cz efmfujoh ui f ufsn t dpssftqpoejoh up a_i ? 1 boe beejoh ui f ufsn t dpssftqpoejoh jefoujdbm b_i(t ofjui fs ui f dpoejujpot- ops ui f dpodmvtjpo di bohf0 Evf up Ui fpsfn 4021- jg ui fsf jt b opodpotubou tpmvujpo- ui fo $\prod_{i=1}^{n} a_i\phi)b_i$+? 1- xi fsf ϕ; $\mathbb{Q})b_1, \ldots, b_n$+⇐ $\mathbb{C}$ jt bo jokfdujwf i pn pn psqi jtn 0

Bttvn joh)j+boe ubl joh jouip dpotjefsbujpo ui bu ui f jefoujuz jt ui f pomz jtpn psqi jtn pwfs $\mathbb{Q}$- xf ffibe $\prod_{i=1}^{n} a_ib_i$? $\prod_{i=1}^{n} a_i\phi)b_i$+? 1- dpousbejdujoh ui f bttvn qujpo0

Bttvn joh)jj+ bhbjo xf vtf ui f gbdu ui bu ϕ ffiyft ui f fmfn fout pg $\mathbb{Q}$0 Yf pcubjo

$$\phi\Big)\prod_{i=1}^{n} a_ib_i\Big(? \prod_{i=1}^{n} a_i\phi)b_i+? 1$$

boe $\prod_{i=1}^{n} a_ib_i$? 1- b dpousbejdujpo0 □

Opx xf ti px ui bu Ui fpsfn 3043 jt b dpotfrvfodf pg Ui fpsfn 40210

Theorem 3.14.)j+ *Suppose that the parameters* $b_1, \ldots, b_s$ *are algebraic numbers and* $b_{s+1}, \ldots, b_n$ *are algebraically independent over* $\mathbb{Q}$, *where* $1 \geq s < n$. *If the parameters* $a_1, \ldots, a_n$ *are algebraic numbers, then*

$$\prod_{i=1}^{n} a_i f)b_ix+? 1 \qquad)2: +$$

has no nonzero additive solution.

)jj+ *Suppose that the parameters* $a_1, \ldots, a_s$ *are algebraic numbers and* $a_{s+1}, \ldots, a_n$ *are algebraically independent over* $\mathbb{Q}$, *where* $1 \geq s < n$. *If the parameters* $b_1, \ldots, b_n$ *are algebraic numbers, then*)2: + *has no nonzero additive solution.*

Proof.)j+ Jg f jt b opo–fsp beejujwf tpmvujpo pg)2: + ui fo cz Ui fpsfn 408 ui fsf fyjtut bo bvupn psqi jtn ϕ pg $\mathbb{C}$ xi jdi jt brtp b tpmvujpo pg)2: +0 Ui vt- $\prod_{i=1}^{n} a_i\phi)b_i$+? 10 Mfu vt bttvn f ui bu fwfsz a_i jt bmhfcsbjd0 Tjodf $b_1, \ldots, b_s$ bsf bmhfcsbjd ovn cfst- ui f bmhfcsbjd joefqfoefodf pg $b_{s+1}, \ldots, b_n$)$s < n$+ jn qmjft ui bu b_n jt usbotdfoefoubm pwfs ui f bmhfcsbjd dmptvsf pg $\mathbb{Q})b_1, \ldots, b_{n-1}$+0 Ui f fybdu tubufn fou boe ui f qsppg dbo cf gpvoe jo]27- Mfn n b 40 02- q0 211'0 Tjodf ϕ jt bo bvupn psqi jtn - tp jt ϕ^{-1}- boe xf dbo sf gpsn vmbuf ui f qsfwjpvt frvbujpo bt $\prod_{i=1}^{n} \phi^{\ \ 1})a_i+b_i$? 10 Ui jt jt b mjofbs dpn cjobujpo pg ui f b_i(t xjui bmhfcsbjd dpfi djfout- tjodf $\phi^{\ \ 1})a+$jt bmhfcsbjd- jg a jt bmhfcsbjd0 Ui jt jt b dpousbejdujpo0

)jj+Ui f qsppg jt tjn jmbs0 Ui f pomz ejggfsfodf jt ui bu xf vtf ϕ jotufbe pg $\phi^{\ \ 1}$- pui fsxjtf ui f bshvn fou jt ui f tbn f0 □

Jg xf qvu a_1 ? b_1 ? 2- xf pcubjo ui f apomz jg(qbsu pg Ui fpsfn 30430 Y f xjmm hfofsbmj{f Ui fpsfn 4025 jo Ui fpsfn 60250

3.3 Existence of solutions of higher degree

Y f tubsu xjui tpn f jn qpsubou jefbt xi jdi xf vtf up jowftujhbuf ui f dbtft xi fo ui f tp. mvujpot bsf hfofsbmj{fe qpmzopn jbmt pg efhsff $k > 2$0 Jo ui jt tfdujpo pvs n bjo sftvmu jt Ui fpsfn 40290

Lemma 3.15. *If $\phi_1, \ldots, \phi_k$ are distinct injective homomorphisms of the field K into $\mathbb{C}$, then there exists an element x / K such that $\phi_i)x+$? $\phi_j)x+$for every $2 \geq i < j \geq k$.*

Proof. Y f vtf joevdujpo po k0 Gps k ? 2 boe k ? 3 ui f tubufn fou jt dmfbs0 Tvqqptf $k > 3$ boe ui f tubufn fou jt usvf gps k 20 Mfu $\phi_1, \ldots, \phi_k$ cf ejtujodu jokfdujwf i pn pn psqi jtn t0 Cz ui f joevdujpo i zqpui ftjt- ui fsf fyjtut bo x / K tvdi ui bu $\phi_1)x+, \phi_2)x+, \ldots, \phi_{k\ \ 1})x+$ bsf ejtujodu0 Jg ui fz bsf ejggfsfou gspn $\phi_k)x+$- ui fo xf bsf epof0 Tvqqptf gps fybn qmf ui bu $\phi_1)x+$? $\phi_k)x+$0 Tjodf ϕ_1 i? ϕ_k, ui fsf jt bo x^∞tvdi ui bu $\phi_1)x^\infty+$i? $\phi_k)x^\infty+$0 Gps fwfsz $i < k$ ui f ovn cfs pg jouf hfst m_i tbujtgzjoh $\phi_i)x$, $m_i x^\infty+$? $\phi_1)x$, $m_i x^\infty+$jt gjojuf0 Ui vt ui fsf sfn bjot b tvjubcmf fmfn fou pg ui f gpsn x , mx^∞0 □

Lemma 3.16. *Let $\phi_1, \ldots, \phi_m$ be distinct injective homomorphisms of the field K into $\mathbb{C}$, and let k be a positive integer. Then there exists an element h / K such that*

$$\int_{j/\!\!\!\,J} \phi_j)h+i? \int_{j'/\!\!\!\,J'} \phi_{j'})h+$$

whenever J and $J^\infty$are distinct multisets of the elements $2, \ldots, m$ containing each of $2, \ldots, m$ at most k times.

Proof. Mfu x cf bt jo Mfn n b 40260 Gps fwfsz n vmujtfu J mfu

$$P_J)r+? \int_{j/\!\!\!\,J} \phi_j)r \quad x+? \int_{j/\!\!\!\,J})r \quad \phi_j)x++$$

gps fwfsz r / $\mathbb{Q}$0 Ui fo P_J jt b qpmzopn jbm pg ui f wbsjbcmf r / $\mathbb{Q}$0 Jg ui f n vmujtfut J, J^∞ bsf ejtujodu- ui fo ui f qpmzopn jbmt $P_J, P_{J'}$ bsf bmtp ejtujodu- cfdbvtf- cz ui f di pjdf pg x- ui f ovn cfst $\phi_j)x+)j$? $2, \ldots, m+$bsf ejtujodu- boe ui vt ui f tfu pg sppu t pg P_J xjui n vmujqmjdjujft jt ejggfsfou gspn ui bu pg $P_{J'}$0

Jg ui f n vmjjtfut J, J^{∞}bsf ejtujodu ui fo $P_J)r+$? $P_{J'})r+$gps bmmcvu b ffjojuf ovn cfs pg r / Q0 Ui fsfgpsf- xf dbo di pptf bo r / Q tvdi ui bu bmmi f wbmvft $P_J)r+$bsf ejtujodu bt J svot ui spvhi ui f qpttjcmf n vmjjtfut0 Ui fo h ? r x tbujtfjft ui f sfrvjsfn fout0 □

Lemma 3.17. *For every field automorphisms $\phi_1, \ldots, \phi_k$ of $\mathbb{C}$, the product $\phi_1 \times \ldots \times \phi_k$ is a solution of*

$$\prod_{i=1}^{n} a_i f)b_i x, \quad c_i y+? \ 1 \qquad)x, y \ / \ \mathbb{C}+ \tag{)31+}$$

if and only if

$$\prod_{i=1}^{n} a_i \int_{j0 J} \phi_j)b_i+ \int_{j/0 J} \phi_{j'})c_i+? \ 1 \tag{)32+}$$

for every $J \leq \}2, \ldots, k|$.

Proof. Mfu

$$H_J \ ? \ \prod_{i=1}^{n} a_i \int_{j0 J} \phi_j)b_i+ \int_{j/0 J} \phi_{j'})c_i+ \qquad)J \to I+.$$

)?$\Leftrightarrow$+Y f efopuf I ? $\}2, \ldots k|$. Jg $\phi_1 \times \ldots \times \phi_k$ jt b tpmvujpo ui fo- bqqmzjoh)31+xjui x ? 2 boe h ? y / $\mathbb{C}$ xf pcubjo

$$\begin{aligned}
1 \ ? \ & \prod_{i=1}^{n} a_i \int_{j=1}^{k})\phi_j)b_i+, \ \phi_j)c_i+\phi_j)y++? \\
? \ & \prod_{i=1}^{n} a_i \times \prod_{J/ I} \int_{j0 J} \phi_j)b_i+ \int_{j/0 J} \phi_{j'})c_i+\phi_{j'})y+? \\
? \ & \prod_{J/ I} \Bigg) \prod_{i=1}^{n} a_i \int_{j0 J} \phi_j)b_i+ \int_{j/0 J} \phi_{j'})c_i+ \sum_{j/0 J} \int \phi_{j'})y+? \\
? \ & \prod_{J/ I} H_J \int_{j/0 J} \phi_{j'})y+.
\end{aligned} \tag{)33+}$$

Y f n bz bttvn f ui bu ui f bvupn psqi jtn t $\phi_1, \ldots, \phi_m$ bsf ejtujodu- boe fbdi pg $\phi_{m+1}, \ldots, \phi_k$ frvbmpof pg $\phi_1, \ldots, \phi_m$. Jg $\overline{J}$ jt b n vmjjtfudpoubjojoh ui f fmfn fou i q_i ujn ft)i ? $2, \ldots, m$+ ui fo xf qvu $H_{\overline{J}}$? H_J- xi fsf $J \to I$ jt tvdi ui bu- gps fwfsz i ? $2, \ldots, m$- ui f dbsejobmjuz pg ui f tfu $\}j \ ; \phi_j \ ? \ \phi_i|$ frvbmt q_i0 Dmfbsmz-)33+dbo cf xsjuufo jo ui f gpsn

$$\prod_{\overline{J}} n_{\overline{J}} H_{\overline{J}} \int_{j/0 \overline{J}} \phi_{j'})y+? \ 1, \tag{)34+}$$

xi fsf $\overline{J}$ svot ui spvhi ui f ejtujodu n vmjjtfut pg tj{f bu n ptu k dpotjtujoh pg fmfn fout $2 \ldots m$- boe ui f dpfi djfou $n_{\overline{J}}$ bsf tvjubcmf qptjujwf joufhfst0 Cz Mfn n b 4027- xf dbo di pptf h tvdi ui bu ui f wbmvft pg ui f qspevdut pg ui ftf jokfdujwf i pn pn psqi jtn t bu h bsf ejtujodu0

Ui f o- bqqmzjoh)34+ x jui y ? $h, h^2, \ldots, h^N$- xf hfu ui f gpmmpx joh tztufn xjui ui f opubujpo $\bigcup_{\overline{J}} \phi_j)h+ ? M_J$;

$$\prod_{\overline{J}} n_{\overline{J}} H_{\overline{J}} \times M_{\overline{J}} ? 1$$

$$\prod_{\overline{J}} n_{\overline{J}} H_{\overline{J}} \times) M_{\overline{J}}+^2 ? 1$$

$$\vdots$$

$$\prod_{\overline{J}} n_{\overline{J}} H_{\overline{J}} \times) M_{\overline{J}}+^N ? 1.$$

Ui jt jt b Wboefsn poef tztufn xi jdi i bt op opousjwjbmtpmvujpo jg N jt bu mfbtu ui f ovn cfs pg n vmujtfut $\overline{J}$ x jui ui f hjwfo qspqfsujft0 Ui jt dpn qmfuft ui f qsppg pg)32+0

)⇒? + Tjodf ϕ_j jt bo bvupn psqi jtn - $\phi_j)b_i x , c_i h+ ? \phi_j)b_i+\phi_j)x+, \phi_j)c_i+\phi_j)h+0$ Ui vt-

$$\prod_{i=1}^{n} a_i \int_{j=1}^{k} \phi_j)b_i x , c_i h+ ? \prod_{i=1}^{n} a_i \int_{j=1}^{k})\phi_j)b_i+\phi_j)x+, \phi_j)c_i+\phi_j)h++ ?$$

$$? \prod_{i=1}^{n} a_i \prod_{J \le \}1,\ldots,k|} \int_{j \in J})\phi_j)b_i+\phi_j)x++ \int_{j' \notin J})\phi_{j'})c_i+\phi_{j'})h++ ?$$

$$? \prod_{J \le \}1,\ldots,k|} \Big) \prod_{i=1}^{n} a_i \times \int_{j \in J} \phi_j)b_i+ \int_{j' \notin J} \phi_{j'})c_i+ \sum \int_{j \in J} \phi_j)x+ \int_{j' \notin J} \phi_{j'})h+ ? 1,$$

cfdbvtf fwfsz ufsn jt 10 Ui vt ui f qspevdu pg $\phi_1 \times \ldots \times \phi_k$ jt b tpmvujpo pg)31+0 □

Theorem 3.18. *For every positive integer k the following are equivalent.*

)j+ *There exists a generalized polynomial of degree k which is a solution of*)31+.

)jj+ *There exist field automorphisms $\phi_1, \ldots, \phi_k$ of $\mathbb{C}$ such that $\phi_1 \times \ldots \times \phi_k$ is a solution of*)31+.

)jjj+ *There exist field automorphisms $\phi_1, \ldots, \phi_k$ of $\mathbb{C}$ such that*

$$\prod_{i=1}^{n} a_i \int_{j \in J} \phi_j)b_i+ \int_{j' \notin J} \phi_{j'})c_i+ ? 1$$

for every $J \le \}2, \ldots, k|$.

Proof.)jj+? ⇔)j+ jt usjwjbm x jui $F_k)x_1, \ldots, x_k+ ? \phi_1)x_1+ \times \ldots \times \phi_k)x_k+$ boe $f_k)x+ ? $ ejbh F_k0)j+? ⇔)jj+; Tvqqptf ui bu ui fsf jt b hfofsbmj{fe qpmzopn jbmtpmvujpo pg)31+ pg efhsff k0 Cz Mfn n b 3041- ui fsf jt b tpmvujpo x i jdi jt b opo–{fsp)hfofsbmj{fe+ n popn jbmpg efhsff k0

Yf sfdbmmi bu S_k efopuft uif tfu pg tpmvujpot pg)31+ efffojfe po K xi jdi bsf n popn jbmt pg efhsff k0 Ui vt S_k i? }1|0 Mfu $f_k)x+?\ F_k)x,\ldots,x+$ cf b tpmvujpo- xi fsf F_k jt opo–fsp- tzn n fusjd boe k.beejujwf0 Mfu $d_1,\ldots,d_k$ / $\mathbb{C}$ cf tvdi ui bu $F_k)d_1,\ldots,d_k+$i? 1. Yf qvu

$$K\ ?\ \mathbb{Q})b_1,\ldots,b_n,c_1,\ldots,c_n,d_1,\ldots,d_k+.$$

xf sfdbmmi bu)$K^{\subsetneq}$+^{k} jt bo Bcfmjbo hspvq voefs n vmujqmjdbujpo cz dppsejobuft0 Tjodf)$K^{\subsetneq}$+^{k} jt dpvoubcmf- tqfdusbmbobmztjt i pmet po ju bddpsejoh up Uifpsfn 30290

Tjodf S_k i? }1|- ju gpmmpxt gspn Mfn n b 404 ui bu $M_k^{\subsetneq}$ i? }1|)gps efffojujpot tff tvctfdujpo 402+0

Cz Mfn n b 403- $M_k^{\subsetneq}$ jt b wbsjfuz po)$K^{\subsetneq}$+^{k}0

Tjodf tqfdusbmbobmztjt i pmet po $M_k^{\subsetneq}$, uifsf jt bo fmfn fou $m)x_1,x_2,\ldots,x_k+$ / $M_k^{\subsetneq}$ xi jdi jt n vmujqmjdbujwf jo fbdi dppsejobuf0 Ui bu jt-

$$m)x_1y_1,x_2y_2,\ldots,x_ky_k+?\ m)x_1,x_2,\ldots,x_k+\times m)y_1,y_2,\ldots,y_k+ \qquad)35+$$

gps fwfsz x_i,y_i / $K^{\subsetneq}$. Cz Mfn n b 407-

$$m)x_1,x_2,\ldots,x_k+?\ \phi_1)x_1+\times\phi_2)x_2+\times\ldots\times\phi_k)x_k+$$

gps fwfsz $x_1,\ldots,x_k,$ / $K^{\subsetneq}$- xi fsf $\phi_1,\ldots,\phi_k$ bsf gjfme bvupn psqi jtn pg $\mathbb{C}$0

)jjj+$\Longrightarrow$)jj+; Ju jt dmfbs cz vtjoh Mfn n b 40280 □

Corollary 3.19. *Suppose that equation*)31+ *satisfies condition*)6+. *Then*)31+ *has a nonconstant solution if and only if there is a k* / $\mathbb{N}$ *and there are automorphisms* $\phi_1,\ldots,\phi_k$ *such that* $\phi_1\times\ldots\times\phi_k$ *is a solution.*

Proof. Jg f jt b opodpotubou tpmvujpo uifo- cz Uifpsfn 3035- ju n vtu cf b hfofsbmj{fe qpmzopn jbm pg efhsff $k>1$0 Uifo xf dbo bqqmz Mfn n b 3041 boe Uifpsfn 40290 □

Remark 3.20. Dpoejujpo)6+ jt ofdfttbsz jo Uifpsfn 3035 boe Dpspmmbsz 402:0 Yf ti pxfe jo Qspqptjujpo 3039 ui bu uif gvodujpobm frvbujpo $f)x+\quad f)3x+?\ 1$ i bt b opodpotubou tpmvujpo bmui pvhi fwfsz hfofsbmj{fe qpmzopn jbm tpmvujpo jt {fsp0 Ui vt- uifsf bsf op bvupn psqi jtn t $\phi_1,\ldots,\phi_k$ tvdi ui bu $\phi_1\times\ldots\times\phi_k$ jt b tpmvujpo- cfdbvtf- dmfbsmz- uif qspevdu $\phi_1\times\ldots\times\phi_k$ jt b hfofsbmj{fe qpmzopn jbm0

Bt xf opufe bgufs Uifpsfn 3038- dpoejujpo)6+ jt bvupn bujdbmmz tbujtffe cz uif gvodujpobm frvbujpot pg uif gpsn

$$\prod_{i=1}^{n} a_if)b_ix\ ,\ \ y+?\ 1 \qquad)36+$$

boe

$$\prod_{i=1}^{n} a_if)b_ix\ ,\ )2\ \ b_i+y+?\ 1. \qquad)37+$$

Ui fsf gpsf- Dpspmmbsz 402: jt usvf gps ui ftf uzqft pg fr vbujpot0 Gps ui ftf fr vbujpot- xf tvqqmfn fou Ui fpsfn 4029 bt gpmmpx t0

Theorem 3.21. *If $\phi_1 \times \ldots \times \phi_k$ is a solution of the functional equation)36+ or)37+, where ϕ_i is an automorphism for every i ? $2,\ldots,k$, then every subproduct of $\phi_1 \times \ldots \times \phi_k$ is also a solution of)36+ or)37+.*

Proof. Ui f fr vbujpot)36+ boe)37+ dbo cf usbotgpsn fe joup fbdi pui fs cz bo joofs mjofbs tvctujuvujpo- boe ui vt ui fz i bwf ui f tbn f tpmvujpot0 Ui fsf gpsf- ju jt fopvhi up dpotjefs)36+0

Cz Mfn n b 4028- $\phi_1 \times \ldots \times \phi_k$ jt b tpmvujpo pg)36+ jg boe pomz jg

$$\prod_{i=1}^{n} a_i \phi_{j_1})b_i+ \ldots \phi_{j_s})b_i+ ? \; 1 \qquad)38+$$

gps fwfsz di pjdf pg ui f joufhfst $1 \geq j_1 < j_2 < \ldots < j_s \geq k$ $)1 \geq s \geq k+$. Ui fsf gpsf- ui f dpoejujpot bsf bvupn bujdbmmz tbujtgjfe gps boz tvcqspevdu- ui vt ui f tubufn fou jt dmfbs0 □

Ui fpsfn 4029 n pujwbuft ui f r vftujpo xi fui fs ps opu ui f fyjtufodf pg b tpmvujpo pg efhsff k jn qmjft ui f fyjtufodf pg tpmvujpot pg ui f gpsn ϕ^k, xi fsf ϕ jt bo bvupn psqi jtn pg $\mathbb{C}$0 Ui f ofyu qspqptjujpo ti px t ui bu ui f botxfs jt ofhbujwf0

Proposition 3.22. *There is a linear functional equation which has a solution of degree two, but has no solution of the form ϕ^2 where ϕ is an isomorphism.*

Proof. Qvu K ? $\mathbb{Q})i+$0 Ui jt gjfme i bt pomz 3 jtpn psqi jtn t- $\phi_1)z+ ? \; z$ boe $\phi_2)z+ ? \; \overline{z}$ 0 Vtjoh Ui fpsfn 40290 ju jt fopvhi up hvbsbouff ui bu;

20 $\prod_{i=1}^{n} a_i$? 1- pui fsxjtf ui fsf jt op opo.usjwjbm tpmvujpo0

30 $\prod_{i=1}^{n} a_i b_i$? 1 boe $\prod_{i=1}^{n} a_i \overline{b_i}$? 1 xi jdi jn qmz ui bu ϕ_1 boe ϕ_2 bsf tpmvujpot pg ui f gvodujpobm fr vbujpo0

40 $\prod_{i=1}^{n} a_i b_i^2$ "? 1- $\prod_{i=1}^{n} a_i \overline{b_i^2}$ "? 1 xi jdi n fbot ui bu ofjui fs ϕ_1^2 ops ϕ_2^2 jt b tpmvujpo0

50 $\prod_{i=1}^{n} a_i \backslash b_i \backslash^2$? 1 xi jdi jn qmjft ui bu $\phi_1\phi_2$ jt b tpmvujpo0

Ju dbo cf fbtjmz ti px o ui bu

$$f)z\,,\;\;2+,\;\; f)z\;\;\;\;2+\;\;\; f)z\,,\;\; i+\;\;\; f)z\;\;\;\; i+ ? \;1$$

jt tvdi bo fr vbujpo0 □

Bt xf qspn jtfe jo ui f mbtu di bqufs- opx xf ti px ui bu ui f usbotmbujpo jowbsjbodf pg ui f tqbdf pg tpmvujpot epft opu jn qmz ui bu ui f gvodujpobm fr vbujpo dbo cf sfevdfe up ui f gpsn)36+0

Proposition 3.23. *There is a functional equation of the type)31+which only has constant solutions, and the points)*b_i, c_i*+/* $\mathbb{C}^2$ *are not collinear.*

Proof. Ju jt fbtz up ffioe- gps fwfsz $n \subset$ 4- joufhfst a_i, b_i boe c_i)i ? $2, \ldots, n$+xjui ui f gpmmpx joh qspqfsujft;

20 $\prod_{i=1}^{n} a_i$? 1-

30 $\prod_{i=1}^{n} a_i b_i^r$ í? 1 gps boz r ? $2, 3, \ldots$-

40 a_i, b_i, c_i tbujtgz dpoejujpo)6+ boe

50 ui f qpjout)b_i, c_i+bsf opu dpmmjofbs0

)Gps fybn qmf- $f)x+$, $f)x$, $y+$ 3 ×$f)3y+$? 1 jt tvdi bo frvbujpo0+ Tjodf ui f jefoujuz jt ui f pomz jolfdujwf i pn pn psqi jtn pg $\mathbb{Q}$- ju gpmmpx t gspn Ui fpsfn 4029 ui bu jg ui fsf jt b opodpotubou tpmvujpo- ui fo ui fsf jt b tpmvujpo x i jdi jt ui f m^{th} qpxfs pg ui f jefoujuz po $\mathbb{Q}$ gps b tvjubcmf m / N0 Tvctujuvujoh x ? 2- y ? 1 xf hfu $\prod_{i=1}^{n} a_i b_i^m$ x i jdi jt opo–fsp0 Ui vt boz tpmvujpo pg ui jt frvbujpo n vtu cf dpotubou0 □

4 Linear functional equations with algebraic parameters

4.1 The space of additive solutions in the algebraic case

Definition 4.1. Mfu K efopuf b tvcgfme pg $\mathbb{C}$0 Xf tbz ui bu b gvodujpo d ; $K \Leftarrow K$ jt b *derivation* jg ju i bt ui f gpmmpxjoh uxp qspqfsujft;

$$d)x \,,\ \ y+?\ d)x+,\ \ d)y+ \quad \text{boe} \quad d)xy+?\ d)x+y\,,\ \ xd)y+ \qquad)39+$$

gps fwfsz x, y / K. Uif gvodujpo d ; $K \Leftarrow \mathbb{C}$ jt b *derivation in the wide sense*- jg ju tbujtgjft)39+gps fwfsz x, y / K.

Lemma 4.2. *Let K be a subfield of $\mathbb{C}$. If d is derivation (in the wide sense) on K, then it can be extended to $\mathbb{C}$ as a derivation (in the wide sense).*

Uif qsppg dbo cf gpvoe jo]27- Uifpsfn 25030B‘0

Pvs n bjo sftvmu jo ui jt tvctfdujpo jt ui f gpmmpxjoh uifpsfn 0

Theorem 4.3. *Let $b_1, \ldots, b_n, c_1, \ldots, c_n$ be algebraic numbers, and put K ? $\mathbb{Q})b_1, \ldots, b_n, c_1, \ldots, c_n$+ Then every additive solution of*

$$\prod_{i=1}^{n} a_i f)b_i x\,,\ \ c_i y+?\ 1 \qquad)x, y\ /\ K+ \qquad)3:+$$

is of the form

$$d_1\phi_1\,,\ \ \times\times\times,\ \ d_k\phi_k,$$

where $d_1, \ldots, d_k$ are complex numbers and $\phi_1, \ldots, \phi_k$; $K \Leftarrow \mathbb{C}$ are injective homomorphisms satisfying

$$\prod_{i=1}^{n} a_i\phi_j)b_i+?\ 1 \text{ and } \prod_{i=1}^{n} a_i\phi_j)c_i+?\ 1 \qquad)41+$$

for every j / $\}2, \ldots, k|$.

Proof. Jg f jt bo beejujwf tpmvujpo pg)3:+po K- ui fo

$$1\ ?\ \prod_{i=1}^{n} a_i f)b_i x\,,\ \ c_i y+?\ \prod_{i=1}^{n} a_i f)b_i x+,\ \ \prod_{i=1}^{n} a_i f)c_i y+$$

gps fwfsz x, y / K0 Cz tvctujuvujoh x ? 1 boe y ? 1 xf hfu $\prod_{i=1}^{n} a_i f)c_i y+?\ 1$ boe $\prod_{i=1}^{n} a_i f)b_i x+?\ 1$- sftqfdujwfmz0 Xf sfdbmm ui bu S_1 efopuft ui f tfu pg beejujwf tpmvujpot pg)3:+po K0 Cz Mfn n b 403- ui f tfu $S_1^{\subsetneq}$? $\}f\backslash_{K^*}$; f / $S_1|$ jt b wbsjfuz po ui f Bcfmjbo hspvq $K^{\subsetneq}$.

Tjodf ui f b_i(t bsf bmhfcsbjd ovn cfst- ui f fffme K jt b ffojuf bmhfcsbjd fyufotjpo pg ui f fffme $\mathbb{Q}$0 Jg $\beta_1,\dots,\beta_N$ jt b cbtjt pg K bt b mjofbs tqbdf pwfs $\mathbb{Q}$ boe jg f ; $K \Leftarrow \mathbb{C}$ jt beejujwf- ui fo

$$f)r_1\beta_1\ ,\ \ \dots,\ \ r_N\beta_N+?\ r_1 f)\beta_1+,\ \ \dots,\ \ r_N f)\beta_N+$$

gps fwfsz $r_1,\dots,r_N$ / $\mathbb{Q}$. Ui fsfgpsf- ui f wbmvft pg f bu boz qpjou pg K bsf efufsn jofe cz ui f wbmvft $f)\beta_1+,\dots,f)\beta_N+$0 Tjodf ui f tfu pg bmmgvodujpot n bqqjoh $\}\beta_1,\dots,\beta_N|$ joup $\mathbb{C}$ i bt ejn fotjpo N ju gpmmpx t ui bu ui f tfu $\mathcal{D}$ pg bmmbeejujwf gvodujpot effofe po K gpsn t b ffojuf ejn fotjpobmmjofbs tqbdf pwfs $\mathbb{C}$. Ui jt jn qmjft ui bu S_1, bt b mjofbs tvctqbdf pg $\mathcal{D}$ jt bmtp ffojuf ejn fotjpobm0 Dpotfrvfoumz- $S_1^{\subsetneq}$ jt b ffojuf ejn fotjpobmusbotmbujpo jowbsjbou mjofbs tqbdf pg gvodujpot effofe po $K^{\subsetneq}$. Cz Ui fpsfn 3025- ju gpmmpx t ui bu fwfsz fmfn fou pg $S_1^{\subsetneq}$ jt b qpmzopn jbmfyqpofoujbmgvodujpo0

Jo pui fs xpset- fwfsz fmfn fou f / $S_1^{\subsetneq}$ dbo cf xsjuufo bt b ffojuf tvn f ? $\prod_{j=1}^{M} p_j \times m_j$, xi fsf $p_1,\dots,p_M$ bsf opo–fsp qpmzopn jbmt boe $m_1,\dots,m_M$ bsf ejtujodu fyqpofoujbmt po $K^{\subsetneq}$. Cz Mfn n b 3026- $m_1,\dots,m_M$ / $S_1^{\subsetneq}$. Ui fsfgpsf- cz Mfn n b 406- m_i dbo cf fyufoefe up $\mathbb{C}$ bt bo bvupn psqi jtn 0 Yf ti bmmefopuf ui f fyufotjpo bmtp cz m_i0 Dmfbsmz- ui f sftusjdujpo pg m_i up K jt bo jokfdujwf i pn pn psqi jtn 0

Yf qspwf ui bu fbdi p_j jt dpotubou0 Tvqqptf ui bu p_1 jt opu dpotubou0 Ui fo- cz Mfn n b 3026- ui fsf jt b opo–fsp beejujwf gvodujpo A po $K^{\subsetneq}$ tvdi ui bu $A \times m_1$ / $S_1^{\subsetneq}$.

Yf qvu d ? $m_1{}^1 \equiv)A \times m_1+$boe $d)1+?$ 1. Yf ti px ui bu d jt b efsjwbujpo jo ui f xjef tfotf0 Tjodf m_1 jt bo bvupn psqi jtn - ju jt bo beejujwf gvodujpo- boe ui fo $m_1{}^1$ jt bmtp beejujwf0 Tjodf $A \times m_1$ / $S_1^{\subsetneq}$- ui vt $A \times m_1$ bmtp beejujwf- boe ui f dpn qptjujpo d jt bmtp beejujwf=ui bu jt- $d)x\ ,\ \ y+?\ \ d)x+,\ \ d)y+$0 Tjodf A jt beejujwf po $K^{\subsetneq}$ xjui sftqfdu up ui f n vmujqmjdbujpo- xf i bwf $A)xy+?\ \ A)x+,\ \ A)y+$gps fwfsz x,y / $K^{\subsetneq}$. Ui fsfgpsf-

$$
\begin{aligned}
d)xy+?\ \ )m_1{}^1 \equiv)A \times m_1+)xy+?\ \ & m_1{}^1)A)xy+\times m_1)xy++? \\
?\ \ & m_1{}^1))A)x+,\ \ A)y++\times m_1)x+m_1)y++? \\
?\ \ & m_1{}^1)A)x+\times m_1)x++\times y\ ,\ \ m_1{}^1)A)y+\times m_1)y++\times x\ ? \\
?\ \ & d)x+y\ ,\ \ d)y+x
\end{aligned}
\qquad)42+
$$

gps fwfsz x,y / $K^{\subsetneq}$. Dmfbsmz- $d)xy+?\ \ d)x+y\ ,\ \ d)y+x$ i pmet jo ui f dbtft x ? 1 ps y ? 1 bt xfmm Ui fsfgpsf- d jt b efsjwbujpo jo ui f xjef tfotf0

Po ui f pui fs i boe- ju jt xfmmlopxo]27- Mfn n b 25020‘)boe ju jt fbtz up di fdl +ui bu po bmhfcsbjd fyufotjpot pg $\mathbb{Q}$ ui f pomz efsjwbujpo jo ui f xjef tfotf jt ui f jefoujdbmmz –fsp gvodujpo0 I px fwfs- ui f gvodujpo A jt opu jefoujdbmmz –fsp boe m_1 jt bo bvupn psqi jtn - ui vt d dboopu cf jefoujdbmmz –fsp0 Ui jt dpousbejdujpo ti px t ui bu p_1 n vtu cf dpotubou- xi jdi dpn qmfuft ui f qsppg0 □

4.2 The space of solutions of higher degree

Opx xf bsf joufsftufe jo ui f tpmvujpot pg)3: +pg bscjusbsz efhsff0 Ui f gpmmpxjoh ui fpsfn hfofsbmj{ft Ui fpsfn 5040

Theorem 4.4. *Let b_i, c_i)i ? 2, ..., n+ be algebraic numbers, and let f be a solution of)3: + defined on K ? $\mathbb{Q}$)$b_1, \dots, b_n, c_1, \dots, c_n$+. If f is a generalized polynomial of degree at most k, then f is the linear combination of products of at most k injective homomorphisms of K which products are also solutions of)3: +.*

Proof. Cz Mfn n b 3041- f jt b tvn pg n popn jbmt pg efhsff bu n ptu k- fbdi jt b tpmvujpo0 Ui fsfgpsf- ju jt fopvhi up qspwf ui bu jg $f_k)x+?$ $F_k)x, \dots, x+$ jt b tpmvujpo- xi fsf F_k jt opo–fsp- tzn n fusjd boe k.beejujwf- ui fo F_k jt ui f mjofbs dpn cjobujpo pg gvodujpot pg ui f gpsn $\phi_1 \times\times\times \phi_k$, xi fsf $\phi_1, \dots, \phi_k$; $K \Leftarrow \mathbb{C}$ bsf jokfdujwf i pn pn psqi jtn t tvdi ui bu ui f ufsn t pg ui f mjofbs dpn cjobujpo bsf tpmvujpot pg)3: +bt xfmm0

Mfu M_k boe $M_k^{\subseteq}$ cf bt jo Mfn n bt 403 boe 4040

Cz Mfn n b 403- $M_k^{\subseteq}$? }$F\backslash_{(K^*)^k}$; F / M_k| jt b wbsjfuz0 Tjodf f_k jt b tpmvujpo- cz Mfn n b 404- F_k / M_k0 Ui fsfgpsf- $M_k^{\subseteq}$ i? }1| 0

Ui f qbsbn fufst b_i(t boe c_i(t bsf bmhfcsbjd ovn cfst- ui vt ui f gjfme K jt b gjojuf bmhfcsbjd fyufotjpo pg ui f gjfme $\mathbb{Q}$0 Mfu $\beta_1, \dots, \beta_N$ cf b cbtjt pg K bt b mjofbs tqbdf pwfs $\mathbb{Q}$. Jg E_k ;)$K^{\subseteq}$+^{k} $\Leftarrow \mathbb{C}$ jt k.beejujwf- ui fo

$$E_k)x, \dots, x+? \; E_k)r_1\beta_1 , \; \dots, \; r_N\beta_N, \dots, r_1\beta_1 , \; \dots, \; r_N\beta_N+?$$
$$? \;)r_1+^k E_k)\beta_1, \dots, \beta_1+, \; \dots, \;)r_N+^k E_k)\beta_N, \dots, \beta_N+?$$
$$? \prod_{(i_1,\dots,i_k)\emptyset I^k} \Big) \int_{s=1}^{k} r_{i_s} \Big(E_k)\beta_{i_1}, \dots, \beta_{i_k}+$$

gps fwfsz $r_1, \dots, r_N$ / $\mathbb{Q}$ boe I ? }$2, \dots, N$| 0

Ui fsfgpsf- ui f wbmvft pg E_k bu boz qpjou pg)$K^{\subseteq}$+^{k} bsf efufsn jofe cz ui f wbmvft $E_k)\beta_1, \dots, \beta_1+, \dots, E_k)\beta_N, \dots, \beta_N+$0 Tjodf ui f tfu pg bmm gvodujpot n bqqjoh })$\beta_1, \dots, \beta_1+, \dots,)\beta_N, \dots, \beta_N+$| joup $\mathbb{C}$ i bt ejn fotjpo N^k- ju gpmmpxt ui bu ui f tfu $\mathcal{F}_k$ pg bmm k.beejujwf gvodujpot efgjofe po)$K^{\subseteq}$+^{k} gpsn t b gjojuf ejn fotjpobm wfdups tqbdf pwfs $\mathbb{C}$. Ui jt jn qmjft ui bu $V^{\subseteq}$, bt b mjofbs tvctqbdf pg $\mathcal{F}_k$ jt bmtp gjojuf ejn fotjpobm0

Dpotfrvfoumz- $M_k^{\subseteq}$ jt b gjojuf ejn fotjpobm usbotmbujpo jowbsjbou- dmptfe mjofbs tqbdf pg gvodujpot efgjofe po)$K^{\subseteq}$+^{k}. Cz Ui fpsfn 3025- ju gpmmpxt ui bu fwfsz fmfn fou pg $M_k^{\subseteq}$ jt b qpmzopn jbm.fyqpofoujbm gvodujpo0

Jo qbsujdvmbs- F_k ? $\prod_{j=1}^{L} P_j \times N_j$, xi fsf $P_1, \dots, P_L$ bsf opo–fsp qpmzopn jbmt boe $N_1, \dots, N_L$ bsf ejtujodu fyqpofoujbmt po)$K^{\subseteq}$+^{k} xjui sftqfdu up ui f n vmujqmjdbujpo0

Y f ti bmmqspwf ui buf wfsz fyqpofoujbmfrfm foupg $M_k^{\subseteq}$ jt pgui f gpsn $\phi_1)x_1+\ldots\phi_k)x_k+\backslash_{(K^*)^k}$ xi fsf ϕ_j jt bo jokfdujwf i pn pn psqi jtn gps fwfsz j / }$2,\ldots,k$| 0 Y f ti bmmbmtp qspwf ui bu fwfsz fyqpofoujbmn popn jbmfrfm foupg $M_k^{\subseteq}$ jt b dpotubou n vmjqmf pg bo fyqpofoujbm0 Ui jt xjmmjn qmz

$$F_k)x_1,\ldots,x_k+?\ \prod_{j=1}^{L} c_j\ \times\phi_1)x_1+\ldots\phi_k)x_k+$$

gps fwfsz $x_1,\ldots,x_k$ / $K^{\subseteq}$. Qvuujoh x_1 ? ... ? x_k ? x xf pcubjo

$$f_k)x+?\ \ F_k)x,\ldots,x+?\ \prod_{j=1}^{L} c_j\ \times\phi_1)x+\ldots\phi_k)x+ \qquad)43+$$

gps fwfsz x / $K^{\subseteq}$. Tjodf)43+jt bmtp usvf gps x ? 1, ui jt xjmmdpn qmfuf ui f qsppg0

Mfu $N)x_1,x_2,\ldots,x_k+$cf bo fyqpofoujbmfrfm foupg $M_k^{\subseteq}$0 Ui fo N jt n vmjqmjdbujwf jo fbdi dppsejobuf- ui vt cz Mfn n b 407-

$$N)x_1,x_2,\ldots,x_k+?\ \ N)x_1,2,\ldots,2+N)2,x_2,\ldots,2+\times\times\times N)2,2,\ldots,x_k+?$$
$$?\ \ m_1)x_1+m_2)x_2+\times\times\times m_k)x_k+$$

gps fwfsz $x_1,\ldots,x_k$ / $K^{\subseteq}$. Vtjoh Mfn n b 406- m_j dbo cf fyufoefe bt bo bvupn psqi jtn pg $\mathbb{C}$0

Opx xf qspwf ui bu fbdi P_j jt dpotubou0

Tvqqptf- f0h0 ui bu P_1 jt opu dpotubou0 Ui fo- cz Mfn n b 3026- ui fsf jt b opo–fsp beejujwf gvodujpo A po)$K^{\subseteq}$+^{k} tvdi ui bu

$$d)x_1,\ldots,x_k+?\ \ A)x_1,\ldots,x_k+\times m_1)x_1+\times\times\times m_k)x_k+/\ V^{\subseteq}.$$

Ui f beejujwjuz pg A n fbot ui bu

$$A)x_1\times y_1,x_2\times y_2,\ldots,x_k\times y_k+?\ \ A)x_1,x_2,\ldots,x_k+,\ \ A)y_1,y_2,\ldots,y_k+ \qquad)44+$$

gps fwfsz $x_1,\ldots,x_n,y_1,\ldots,y_n$ / $K^{\subseteq}$0 Y f qspwf ui bu jo ui jt dbtf $d \subseteq 1$ xi jdi jt b dpousb. ejdujpo0

Ui fo $d)x_1,2,\ldots,2+?\ \ A)x_1,2,\ldots,2+m_1)x_1+$jt b qpmzopn jbmfyqpofoujbmpgui f dbtf pg Ui fpsfn 504- ui fsfgpsf $d)x_1,2,\ldots,2+?$ 1 boe tjodf $m_1)x_1+?$ 1- $A)x_1,2,\ldots,2+?$ 10 Ui jt jt usvf gps fwfsz dppsejobuf0 Vtjoh)44+ xf pcubjo ui bu

$$A)x_1,x_2,\ldots,x_k+?\ \ A)x_1,2,\ldots,2+,\ \ \times\times\times,\ \ A)2,2,\ldots,x_k+?\ \ 1$$

gps fwfsz $x_1,x_2,\ldots,x_k$ / $K^{\subseteq}$- ui fsfgpsf d n vtu cf –fsp0 $\square$

Remark 4.5. Jo Mfnnb 4028 xf tipxfe uibu jg $\phi_1,\ldots,\phi_k ; K \Leftarrow \mathbb{C}$ bsf jokfdujwf ipnp-npsqijtnt uifo uif qspevdu $\phi_1 \times\times\times \phi_k$ jt b tpmvujpo pg)3:+ jg boe pomz jg

$$\prod_{i=1}^{n} a_i \int_{j\in J} \phi_j)b_i+ \int_{j\notin J} \phi_{j'})c_i+? \ 1 \qquad)45+$$

gps fwfsz $J \leq \}2,\ldots,k|$ 0

Jo uif tqfdjbm dbtf xifo uif gvodujpobm frvbujpo dbo cf xsjuufo pg uif gpsn

$$\prod_{i=1}^{n} a_i f)b_i x ,\ \ y+? \ 1 \qquad)46+$$

uif tjuvbujpo jt tmjhiumz tjnqmfs0 Cz Uifpsfn 4032- jg xf ublf uif jokfdujwf ipnpnpsqijtnt tpmvujpot pg)46+ uifo gpsnjoh uif qspevdut bu npt n 3 pg uifn hjwft b cbtjt pg uif tqbdf pg uif tpmvujpot0 Gvsuifsnpsf- xf tipxfe jo Uifpsfn 4032 uibu uif gbdu uibu f jt b tpmvujpo pg)46+ boe ju jt b qspevdu pg jokfdujwf ipnpnpsqijtnt jnqmjft uibu fwfsz tvcqspevdu pg f jt bmtp tpmvujpo pg)46+0

4.3 Trivial functional equations

Jo uijt tfdujpo xf qsftfou bopuifs dmbtt pg frvbujpot tvdi uibu uif tfu pg jut beejujwf tpmvujpot jt tqboofe cz uiptf tpmvujpot xijdi bsf jokfdujwf ipnpnpsqijtnt0

Xf tbz uibu uif frvbujpo)3:+ jt usjwjbm jg fwfsz beejujwf gvodujpo jt b tpmvujpo pg uif gvodujpobm frvbujpo0 Fybnqmft pg usjwjbm frvbujpot bsf

$$4f)x,\ \ \overline{3}y+ \ \ f)7x,\ )4\ \overline{3},\ 6\ \overline{4}y+,\ 6f)x,\ \ \overline{4}y+? \ 1$$

boe

$$3f)x,\ y+ \ \ f)x,\ 3y+ \ \ 3f)x,\ \pi y+,\ \ f)x,\ 3\pi y+? \ 1.$$

Xf qspwf uibu uif usjwjbm frvbujpot bmtp ibwf uif qspqfsuz uibu uif tfu pg uifjs beejujwf tpmvujpot jt tqboofe cz uiptf tpmvujpot xijdi bsf jokfdujwf ipnpnpsqijtnt0 Uijt bnpvout up tipx uibu uif tfu pg bmm beejujwf gvodujpot jt tqboofe cz jokfdujwf ipnpnpsqijtnt0

Theorem 4.6. *The variety $V_{\mathbb{C}}$ generated by the automorphisms of $\mathbb{C}$ contains every additive function.*

Proof. Uif wbsjfuz $V_{\mathbb{C}}$ dpoubjot uif dmptvsf pg uif tfu pg mjofbs dpncjobujpot pg uif bvupnpsqijtnt pg $\mathbb{C}$. Uijt nfbot uibu xifofwfs $f ; \mathbb{C} \Leftarrow \mathbb{C}$ jt tvdi uibu- gps fwfsz $y_1, y_2, \ldots, y_n$ / $\mathbb{C}$ boe $\varepsilon > 1$ uifsf bsf bvupnpsqijtnt $\phi_1, \phi_2, \ldots, \phi_M$ tbujtgzjoh

$$\backslash f)y_j+ \ \ \prod_{m=1}^{M} c_i \times \phi_m)y_j+\backslash < \varepsilon$$

gps fwfsz j ? $\}2,\ldots,n|$- ui fo f / $V_{\mathbb{C}}$. Yf ti px ui bufwfsz beejujwf gvodujpo po $\mathbb{C}$ i bt ui jt qspqfsuz0

Ju jt fopvhi up qspwf ui bujg K jt ffiojufm hfofsbufe- ui fo ui f wbsjfuz V_K- hfofsbufe cz ui f jolfdujwf i pn pn psqi jtn t pg K joup $\mathbb{C}$ dpoubjot fwfsz beejujwf gvodujpo po K0 Joeffe- gps hjwfo qpjout $y_1, y_2, \ldots, y_n$ / $\mathbb{C}$- xf n bz ubl f ui f tvcfffm K hfofsbufe cz ui ftf qpjout0 Jgxf dbo hvbsbouff ui bu ui f sftusjdujpo pgb hjwfo beejujwf gvodujpo f up K jt jo ui f wbsjfuz V_K gps fwfsz ffiojufm hfofsbufe tvcfffm K pg $\mathbb{C}$- ui fo- cz Qspqptjujpo 405- $V_{\mathbb{C}}$ dpoubjot ui f gvodujpo f0

Yf xjmqspwf ui f gpmpxjoh tuspohfs tubufn fou cz joevdujpo po ui f ovn cfs pghfofsb. upst pg K ; jgb gvodujpo f / V_K boe ui f qpjout $y_1, y_2, \ldots, y_n$ / K bsf hjwfo- ui fo ui fsf bsf jolfdujwf i pn pn psqi jtn t $\phi_1, \phi_2, \ldots, \phi_M$ boe ui fsf bsf dpn qmfy ovn cfst $c_1, \ldots, c_M$ tvdi ui bu

$$f)y_j+? \prod_{m=1}^{M} c_m \times \phi_m)y_j+$$

gps fwfsz j ? $\}2,\ldots,n|$ 0 Ju jt dmfbs ui bu ui f tubufn fou jt usvf gps K ? $\mathbb{Q}$- tjodf fwfsz beejujwf gvodujpo po $\mathbb{Q}$ jt pgui f gpsn $c \times$Je- xi fsf Je efopuft ui f jefoujuz gvodujpo pg $\mathbb{Q}$0 Mfu vt bttvn f ui bu ui f tubufn fou jt usvf gps b ffiojufm hfofsbufe K0 Yf qspwf ui f tubufn fou gps $K)\beta+$ boe gps $K)t+$; xi fsf β jt bo bmhfcsbjd ovn cfs- boe t jt b usbotdfoefoubmovn cfs0

Case 1: β jt bo bmhfcsbjd ovn cfs- boe $\backslash K)\beta+; K\backslash ?\ n$0 Ui fo

$$K)\beta+? \}a_0\,,\ a_1\beta\,,\ \times\times\times,\ a_n\ {}_1\beta^{n\ 1}\,;a_i\ /\ K\ )i\ ?\ 1,\ldots,n\quad 2+|\,.$$

Mfu ui f beejujwf gvodujpo f ; $K)\beta+\Leftarrow \mathbb{C}$ boe ui f ovn cfst $y_1, \ldots, y_N$ / $K)\beta+$cf hjwfo0

Tjodf fwfsz y_j jt b tvn pgufsn t pgui f gpsn $b \times \beta^i$ $)b$ / $K+$ ui fsfgpsf- cz beejujwjuz- ju jt fopvhi up sfqsftfou f bu ui f qpjout $b_{i,j} \times \beta^i$ gps fwfsz $b_{i,j}$ / K $)j$? $2,\ldots,N$, i ? $1,\ldots,n\quad 2+$0 Gps fwfsz ffiy i- ui f gvodujpo $f)x \times \beta^i+$ jt bo beejujwf gvodujpo po K0 Ui vt- cz ui f joevdujpo i zqpui ftjt- ui fsf bsf jolfdujwf i pn pn psqi jtn t $\phi_1, \ldots, \phi_M$ pg K joup $\mathbb{C}$, boe ui fsf bsf dpn qmfy ovn cfst $c_{i,m}$ tvdi ui bu

$$f)b_{i,j} \times \beta^i+? \prod_{m=1}^{M} c_{i,m} \times \phi_m)b_{i,j}+ \qquad)47+$$

gps fwfsz i ? $1,\ldots,n\quad 2$ boe j ? $2,\ldots,N$0 Yf offe up sfqsftfou fwfsz ufsn po ui f sjhi u i boe tjef pg)4: +cz tpn f mjofbs dpn cjobujpo pg wbmvft pgjolfdujwf i pn pn psqi jtn t pg $K)\beta+$ bu ui f qpjou $b_{i,j} \times \beta^i$. Mfu ui f sppu t pgui f n jojn bmqpmzopn jbmpg β cf β ? $\beta_0, \ldots, \beta_n\ {}_1$. Yf qvu

$$\psi_{m,k})a_0\,,\ a_1\beta\,,\ \times\times\times,\ a_n\ {}_1\beta^{n\ 1}+?\ \phi_m)a_0+,\ \phi_m)a_1+\beta_k\,,\ \times\times\times,\ \phi_m)a_n\ {}_1+\beta_k^{n\ 1}$$

gps fwfsz $a_0, \ldots, a_n \;\; 1$ / K boe k ? $1, \ldots, n \;\;$ 20 Ui fo $\psi_{m,k}$ jt bo jolfdujwf i pn pn psqi jtn pg $K)\beta+$joup $\mathbb{C}$ gps fwfsz m boe k. Yf ti px ui bu ui fsf bsf ovn cfst $x_{m,k}$ tvdi ui bu

$$f)b_{i,j}\beta^i+? \prod_{m=1}^{M}\prod_{k=0}^{n\;\;1} x_{m,k}\psi_{m,k})b_{i,j}\beta^i+\!\!= \qquad)48+$$

ui bu jt-

$$f)b_{i,j}\beta^i+? \prod_{m=1}^{M}\prod_{k=0}^{n\;\;1} x_{m,k}\phi_m)b_{i,j}+\times\beta_k^i.$$

Cz)47+ ju jt fopvhi up ffioe $x_{m,k}$ tbujtgzjoh ui f frvbujpot

$$c_{i,m} \times\phi_m)b_{i,j}+? \prod_{k=0}^{n\;\;1} x_{m,k}\phi_m)b_{i,j}+\times\beta_k^i$$

gps fwfsz i ? $1, \ldots, n \;\;$ 2, j ? $2, \ldots, N$ boe m ? $2, \ldots, M$. Ui bu jt- $x_{m,k}$ i bt up tbujtgz

$$c_{i,m} ? \prod_{k=0}^{n\;\;1} x_{m,k} \times\beta_k^i \qquad)49+$$

gps fwfsz i ? $1, \ldots, n \;\;$ 2 boe m ? $2, \ldots, M$. Tjodf ui f efufsn jobou pg ui f tztufn pg frvbujpot)49+jt opo–fsp)Wboefsn poef+gps fwfsz ffiyfe m, ui ftf tztufn t bsf tpmwbcmf0 Jg $x_{m,k}$ jt b tpmvujpo-)48+ti px t ui bu f jt sfqsftfoufe po ui f tfu $\}b_{i,j}\beta^i|$ bt b mjofbs dpn cjobujpo pg jolfdujwf i pn pn psqi jtn t0

Case 2: Jg t jt b usbotdfoefoubmovn cfs- ui fo fwfsz fmfn fou pg $K)t+$jt b sbujpobmgvodujpo pg ui f wbsjbcmf t xjui dpfı djfout gspn K0

Mfu f ; $K)t+\Leftarrow \mathbb{C}$ cf bo beejujwf gvodujpo0 Mfu ui f sbujpobmgvodujpot $y_1, \ldots, y_n$ / $K)t+$ cf hjwfo0 Mfu $q)t+$cf b dpn n po n vmujqmf pg ui f efopn jobupst pg ui f sbujpobmgvodujpot y_i.

Tjodf fwfsz y_j jt b tvn pg ufsn t pg ui f gpsn $b \times t^i/q)t+)b$ / $K+$ ui fsfgpsf- cz beejujwjuz- ju jt fopvhi up sfqsftfou f bu ui f qpjout $b_{i,j} \times t^i/q)t+$gps fwfsz $b_{i,j}$ / K)j ? $2, \ldots, N$, i ? $1, \ldots, n \;\;$ 2+0 Gps fwfsz ffiy i- ui f gvodujpo $f)x \times t^i/q)t++$jt bo beejujwf gvodujpo po K0 Ui vt- cz ui f joevdujpo i zqpui ftjt- ui fsf bsf jolfdujwf i pn pn psqi jtn t $\phi_1, \ldots, \phi_M$ pg K joup $\mathbb{C}$, boe ui fsf bsf dpn qmfy ovn cfst $c_{i,m}$ tvdi ui bu

$$f)b_{i,j} \times t^i/q)t++? \prod_{m=1}^{M} c_{i,m} \times\phi_m)b_{i,j}+ \qquad)4:+$$

gps fwfsz i ? $1, \ldots, n \;\;$ 2 boe j ? $2, \ldots, N$0 Yf offe up sfqsftfou fwfsz ufsn po ui f sjhi u i boe tjef pg)47+cz tpn f mjofbs dpn cjobujpo pg wbmvft pg jolfdujwf i pn pn psqi jtn t pg $K)t+$bu ui f qpjout $b_{i,j} \times t^i/q)t+$ Mfu $t_0, t_2, \ldots, t_{n\;\;1}$ cf bmhfcsbjdbmmz joefqfoefou fmfn fout pwfs K- boe qvu

$$\psi_{m,k})r)t++? \;)\phi_m \equiv r+)t_k+$$

gps fwfsz r / $K)t+$, m ? $2,\ldots,M$ boe k ? $1,\ldots,n$ 20 Ui fo $\psi_{m,k}$ jt bo jokfdujwf i pn pn psqi jtn pg $K)t+$joup $\mathbb{C}$ gps fwfsz m boe k. Y f ti px ui bu ui fsf bsf ovn cfst $x_{m,k}$ tvdi ui bu

$$f)b_{i,j}t^i/q)t++? \prod_{m=1}^{M}\prod_{k=0}^{n\ 1} x_{m,k}\psi_{m,k})b_{i,j}t^i/q)t++= \qquad)51+$$

ui bu jt-

$$f)b_{i,j}t^i/q)t++? \prod_{m=1}^{M}\prod_{k=0}^{n\ 1} x_{m,k}\phi_m)b_{i,j}+\times\frac{t_k^i}{\phi_m)q)t_k++}.$$

Cz)47+ ju jt fopvhi up ffoe $x_{m,k}$ tbujtgzjoh ui f fr vbujpot

$$c_{i,m}\ \times\phi_m)b_{i,j}+? \prod_{k=0}^{n\ 1} x_{m,k}\phi_m)b_{i,j}+\times\frac{t_k^i}{\phi_m)q)t_k++} \qquad)52+$$

gps fwfsz i ? $1,\ldots,n$ 2, j ? $2,\ldots,N$ boe m ? $2,\ldots,M$. Mfu $z_{m,k}$ tbujtgz

$$c_{i,m} ? \prod_{k=0}^{n\ 1} z_{m,k}\ \times t_k^i \qquad)53+$$

gps fwfsz i ? $1,\ldots,n$ 2 boe m ? $2,\ldots,M$. Tjodf ui f efufsn jobou pg ui f tztufn pg fr vbujpot)53+jt opo–fsp)Wboefsn poef+gps fwfsz ffiyfe m- ui ftf tztufn t bsf tpmwbcmf0 Jg $z_{m,k}$ jt b tpmvujpo- ui fo qvu $x_{m,k}$? $z_{m,k}\ \times\phi_m)q)t_k++$. Ui fo)51+ti px t ui bu f jt sfqsftfoufe po ui f tfu $\}b_{i,j}t^i/q)t+\|$ bt b mjofbs dpn cjobujpo pg jokfdujwf i pn pn psqi jtn t0

Ui jt dpn qmfuft ui f qsppg0 $\square$

4.4 Further developments

Ui fpsfn 507 n jhi u tvhhftu ui bu jg ui fsf bsf joffojufmz n boz jokfdujwf i pn pn psqi jtn t xi jdi bsf tpmvujpot pg)3: + ui fo ui f wbsjfuz hfofsbufe cz ui ftf jokfdujwf i pn pn psqi jtn t dpoubjot fwfsz beejujwf tpmvujpo bt xfmm0 Opx xf ti px ui bu ui jt jt opu usvf jo hfofsbmbt Ui fpsfn 508 ti px t cfmpx 0

Theorem 4.7. *Let $K \to \mathbb{C}$ be a field which contains a transcendental number. Then there exists a linear functional equation*

$$\prod_{i=1}^{n} a_i f)b_i x\ ,\ \ y+?\ 1$$

such that b_i / K for every i ? $2,\ldots,n$, and there exists an additive solution on K which is not contained by the variety generated by the injective homomorphisms which are solutions.

Proof. Mfu vt tvqqptf ui bu t / K jt b usbotdfoefoubmovn cfs boe $a_1,\dots,a_n$ bsf opo–fsp dpn qmfy ovn cfst tbujtgzjoh

$$\prod_{i=1}^{n} a_i \ ?\ 1,\ \prod_{i=1}^{n} a_i t^i \ ?\ 1,\ \prod_{i=1}^{n} a_i i t^{i\ \ 1} \ ?\ 1. \qquad)54+$$

Y f xjmmqspwf ui bu jo ui jt dbtf ui f gvodujpobmfr vbujpo

$$\prod_{i=1}^{n} a_i f)t^i x\ ,\ \ y+?\ 1 \qquad)55+$$

tbujtfff t ui f sfr vjsfn fout0

Fwfsz fmfn fou pg $\mathbb{Q})t+$ jt b sbujpobmgvodujpo xjui sbujpobmdpfı djfout pg ui f wbsjbcmf t0 Ui f pqfsbujpo d_0 ? $\frac{\partial}{\partial t}$ jt b xfmmefffiofe efsjwbujpo po $\mathbb{Q})t+$0 Ju gpmmpx t gspn Mfn n b 50B ui bu xf dbo fyufoe d_0 gspn $\mathbb{Q})t+$ up K bt b efsjwbujpo0 Ui fo d_0 jt b tpmvujpo pg)55+0 Joeffe-

$$\begin{aligned} &\prod_{i=1}^{n} a_i)d_0)t^i x\ ,\ \ y++?\ \prod_{i=1}^{n} a_i d_0)t^i +x\ ,\ \prod_{i=1}^{n} a_i t^i d_0)x+,\ \prod_{i=1}^{n} a_i d_0)y+? \\ &?\ \prod_{i=1}^{n} a_i i t^{i\ \ 1} d_0)t+x,\ \prod_{i=1}^{n} a_i t^i d_0)x+,\ \prod_{i=1}^{n} a_i d_0)y+?\ 1 \end{aligned} \qquad)56+$$

cz)54+0

Mfu V efopuf ui f wbsjfuz hfofsbufe cz ui ptf jokfdujwf i pn pn psqi jtn t xi jdi bsf tpmv. ujpot pg)55+ po K.

Tvqqptf ϕ / V jt bo jokfdujwf i pn pn psqi jtn 0 Ui fo- cz)54+ xf i bwf $\prod_{i=1}^{n} a_i \phi)t+^i$? 1, boe ui vt $\phi)t+$ jt b sppu pg ui f qpmzopn jbm $p)x+$? $\prod_{i=1}^{n} a_i x^i$. Mfu $u_1,\dots,u_k$ cf ui f ejtujodu sppu t pg p. Tvqqptf $\phi)t+?\ u_i$. Jg ψ / V jt bopui fs jokfdujwf i pn pn psqi jtn xjui $\psi)t+?\ u_i$, ui fo sftusjdujpot pg ϕ boe ψ up $\mathbb{Q})t+$ dpjodjef0 Ui jt fbtjmz jn qmjft ui bu ui f gbn jmz pg sftusjdujpot W ? $\}g\backslash_{\mathbb{Q}(t)}\ ;\ g$ / $V|$ jt ffojuf ejn fotjpobm0 Ju jt dmfbs ui bu jg $\phi_1,\dots,\phi_k$ bsf ffiyfe jokfdujwf i pn pn psqi jtn t tvdi ui bu $\phi_j)t+?\ u_j$)j ? $2,\dots,k$+- ui fo ui f sftusjdujpot $\phi_j\backslash_{\mathbb{Q}(t)}$ gpsn b cbtjt pg W bt b mjofbs tqbdf0

Tvqqptf d_0 / V. Ui fo ui fsf bsf dpn qmfy ovn cfst $c_1,\dots,c_k$ tvdi ui bu

$$\prod_{j=1}^{k} c_j \phi_j)w+?\ d_0)w+ \qquad)57+$$

gps fwfsz w / $\mathbb{Q})t+$. Bqqmzjoh)57+ xjui w ? t^i boe vtjoh $d_0)t^i+?\ i \times t^{i\ \ 1}$, xf ffioe

$$\prod_{j=1}^{k} c_j \phi_j)t^i+?\ \prod_{j=1}^{k} c_j u_j^i \ ?\ i \times t^{i\ \ 1}$$

gps fwfsz i ? $2,3,\dots$. N vmujqmzjoh cz z^i boe tvn n joh gps i ? $2,3,\dots$ xf pcubjo

$$\prod_{j=1}^{k} c_j \prod_{i=1}^{\in} u_j^i z^i \ ?\ \prod_{i=1}^{\in} i \times t^{i\ \ 1} z^i,$$

bt bo frvbujpo cfuxffo gpsn bmqpxfs tfsjft0 I pxfwfs- gps z tn bmmfopvhi cpui qpxfs tfsjft bsf dpowfshfou- boe xf pcubjo

$$\prod_{j=1}^{k} c_j \frac{u_j z}{2 \quad u_j z} \; ? \; \frac{\partial}{\partial t} \times \frac{2}{2 \quad tz} \; ? \quad \frac{z}{)2 \quad tz+^2} \qquad)58+$$

gps fwfsz $\backslash z\backslash < \delta 0$ Uif gvodujpot xbt effifoe xjui qpxfs tfsjft boe uif joufstfdujpo pg epn bjot pg dpowfshfodf dpoubjo b tfu ui bu i bt b opo.jtpmbufe qpjou0 Uiftf jn qmjft ui bu uif gvodujpot bsf frvbmjo uif vojpo pg uif epn bjot vtjoh Vojrvfoftt Qsjodjqmf0 Uijt jn qmjft ui bu frvbujpo)58+jt bo jefoujuz po uif xi pmf dpn qmfy qmbof- tjodf uif mbtu ufsn pg frvbujpo)58+jt b n fspn psqi jd gvodujpo po uif xi pmf tqbdf xjui b qpmf tjohvmbsjuz jo $z \; ? \; 2/t0$ Uifsfgpsf-

$$\prod_{j=1}^{k} c_j \frac{)2 \quad tz+^2 u_j z}{2 \quad u_j z} \; ? \quad z$$

i pmet gps fwfsz $z \; ? \; 2/t$ boe jg $z \Leftarrow \frac{1}{t}$ uif mfgu tjef ufoet up 1- cvu uif sjhi u jt opu0

Uijt dpousbejdujpo ti pxt ui bu $d_0 \; / \; V$, xi jdi dpn qmfuft uif qsppg0 □

Example 4.8. Dpotjefs uif gvodujpo frvbujpo

$$f)t^2 x\,,\ \ y+\ \ 3tf)tx\,,\ \ y+,\ \ t^2 f)x\,,\ \ y+\ \ )t\ \ 2+^2 f)y+?\ 1, \qquad)59+$$

xifsf t jt usbotdfoefoubm0 Uifsf fyjtut pomz pof jokfdujwf i pn pn psqi jtn tpmvujpo po $K \; ? \; \mathbb{Q})t+0$ Joeffe- tvctujuvujoh ϕ up)59+ xf pcubjo $)\phi)t+ \quad t+^2 \; ? \; 1$ xi jdi jn qmjft ui bu ϕ n vtu cf uif jefoujuz po $K0$ Po uif puifs i boe- $\partial/\partial t$ jt b tpmvujpo po $K0$

5 Spectral synthesis in varieties of additive functions

5.1 Differential operators

Gjstu xf sfdbmmi f efffbjujpo pg b efsjwbujpo0

Gps fwfsz fffm K- b *derivation (on K)* jt b n bq d ; $K \Leftarrow K$ tvdi ui bu $d)x$, $y+?$ $d)x+$, $d)y+$boe $d)xy+?$ $d)x+\times y$, $d)y+\times x$ gps fwfsz x, y / K0 Ju jt xfml opxo ui bu jg d jt b efsjwbujpo)po $K+$boe L jt b fffm dpoubjojoh K- ui fo d dbo cf fyufoefe up L bt b efsjwbujpo)po $L+$0)Tff]27- Ui fpsfn 2503Œ‘+

Tvqqptf ui bu ui f dpn qmfy ovn cfst $t_1, \ldots, t_n$ bsf bmhfcsbjdbmmz joefqfoefou pwfs $\mathbb{Q}$0 Ui f fmfn fout pg ui f fffm $\mathbb{Q})t_1, \ldots, t_n+$bsf ui f sbujpobmgvodujpot pg $t_1, \ldots, t_n$ xjui sbujpobm dpfi djfout0 Cz b *differential operator on* $\mathbb{Q})t_1, \ldots, t_n+$xf n fbo bo pqfsbups pg ui f gpsn

$$D \; ? \; \prod c_{i_1,\ldots,i_n} \times \frac{\partial^{i_1+***+i_n}}{\partial t_1^{i_1} \times\times\times \partial t_n^{i_n}}, \qquad)5:+$$

xi fsf $\partial/\partial t_i$ bsf ui f vtvbmqbsujbmefsjwbujwft- ui f tvn jt ffojuf- jo fbdi ufsn ui f dpfi djfou jt b dpn qmfy ovn cfs- boe ui f fyqpofout $i_1, \ldots, i_n$ bsf opoofhbujwf joufhfst0 Jg i_1 ? . . . ? i_n ? 1- ui fo cz $\partial^{i_1+***+i_n}/\partial t_1^{i_1} \times\times\times \partial t_n^{i_n}$ xf n fbo ui f jefoujuz pqfsbups po $\mathbb{Q})t_1, \ldots, t_n+$0 Ui f efhsff pg ui f ejfifsfoujbmpqfsbups D jt ui f n byjn vn pg ui f ovn cfst i_1 , . . . , i_n tvdi ui bu $c_{i_1,\ldots,i_n}$ > 10

Ju jt pcwjpvt ui bu $\partial/\partial t_i$ jt b efsjwbujpo po $\mathbb{Q})t_1, \ldots, t_n+$gps fwfsz i ? 2, . . . , n0 Ui fsf. gpsf- fwfsz ejfifsfoujbmpqfsbups po $\mathbb{Q})t_1, \ldots, t_n+$jt ui f mjofbs dpn cjobujpo xjui dpn qmfy dpfi djfout pg ffojufmz n boz n bqt pg ui f gpsn $d_1 \equiv \ldots \equiv d_k$- xi fsf $d_1, \ldots, d_k$ bsf efsjwbujpot po $\mathbb{Q})t_1, \ldots, t_n+$ Ui jt pctfswbujpo n pujwbuft ui f gpmmpxjoh efffojujpo0

Definition 5.1. Mfu K cf b tvcfffm pg $\mathbb{C}$0 Yf tbz ui bu ui f n bq D ; $K \Leftarrow \mathbb{C}$ jt b *differential operator* po K- jg D jt ui f mjofbs dpn cjobujpo- xjui dpn qmfy dpfi djfout- pg ffojufmz n boz n bqt pg ui f gpsn $d_1 \equiv \ldots \equiv d_k$- xi fsf $d_1, \ldots, d_k$ bsf efsjwbujpot po K0 Jg k ? 1 ui fo xf joufsqsfu $d_1 \equiv \ldots \equiv d_k$ bt ui f jefoujuz gvodujpo po K0

Opuf ui bu jg $K \to L \to \mathbb{C}$ bsf fffmet boe D jt b ejfifsfoujbmpqfsbups po K- ui fo D dbo cf fyufoefe up L bt b ejfifsfoujbmpqfsbups0 Ui jt jt dmfbs gspn ui f gbdu ui bu fwfsz efsjwbujpo dbo cf fyufoefe gspn K up L0

Yf ti px ui bu jg K ? $\mathbb{Q})t_1, \ldots, t_n+$ ui fo ui f uxp efffojujpot pg ejfifsfoujbmpqfsbupst dpjodjef0 Bduvbmmz- n psf jt usvf0

Proposition 5.2. *Let K be a subfield of $\mathbb{C}$, and suppose that the elements $t_1, \ldots, t_n$ / K are algebraically independent over $\mathbb{Q}$. If D is a differential operator on K according to Definition 5.1, then the restriction of D to $\mathbb{Q})t_1, \ldots, t_n+$is of the form)5:+.*

Proof. Qvu $\mathbb{Q})t_1,\ldots,t_n+?\ F$- boe mfu $\mathcal{E}$ efopuf ui f tfu pg bmm gvodujpot efgjofe po F ui bu dbo cf sfqsftfoufe jo ui f gpsn)5: +0 Ju jt fopvhi up ti px ui bu jg $d_1,\ldots,d_k$ bsf efsjwbujpot)po K+ ui fo ui f sftusjdujpo pg $d_1 \equiv \ldots \equiv d_k$ up F cfmpoht up $\mathcal{E}$0 Y f qspwf ui jt cz joevdujpo po k0 Gjstu x f opuf ui bu jg d jt b efsjwbujpo boe $d)t_i+?\ \alpha_i\ )i\ ?\ 2,\ldots,n+$ ui fo

$$d)x+?\ \prod_{i=1}^{n} \alpha_i \times \frac{\partial x}{\partial t_i} \qquad)61+$$

gps fwfsz $x\ /\ F$0 Joeffe-)61+ dbo cf fbtjmz di fdlfe gjstu gps fwfsz $x\ /\ \mathbb{Q}]t_1,\ldots,t_n`$ boe ui fo gps fwfsz $x\ /\ F$0 Ui fsfgpsf- ui f tubufnfou jt usvf gps $k\ ?\ 20$

Mfu $k > 2$- boe tvqqptf ui f tubufnfou jt usvf gps $k\quad 20$ Mfu $d_1,\ldots,d_k$ cf efsjwbujpot)po K+0 Ui fo- cz ui f joevdujpo i zqpui ftjt- ui f nbq $g\ ?\ d_2\equiv\ldots\equiv d_k$ sftusjdufe up F cfmpoht up $\mathcal{E}$0 Mfu $g\ ?\ \prod_{j=1}^N c_j g_j$- x i fsf fbdi g_j jt pg ui f gpsn $\partial^{i_1+\ldots+i_n}/\partial t_1^{i_1}\times\times\times\partial t_n^{i_n}$. Fyufoe d_1 up $\mathbb{C}$ bt b efsjwbujpo0 Ui fo

$$d_1\equiv g\ ?\ d_1\equiv\Big)\prod_{j=1}^N c_j g_j\Big(\ ?\ \prod_{j=1}^N)d_1)c_j+\times g_j\ ,\ \ c_j\times)d_1\equiv g_j++$$

Opx ui f tubufnfou $)d_1\equiv g+_{|F}\ /\ \mathcal{E}$ gpmmpx t gspn)61+ x i fo bqqmjfe up $d\ ?\ d_1$0 □

Jo ui f tfrvfm x f ti bmm efopuf cz j ui f jefoujuz gvodujpo efgjofe po $\mathbb{C}$0

Theorem 5.3. *Let K be a subfield of $\mathbb{C}$, and let D be a differential operator on K. Then D/j is a polynomial on $K^{\subsetneq}$.*

Proof. Ju jt fopvhi up ti px ui bu jg $d_1,\ldots,d_n$ bsf efsjwbujpot po K- ui fo $)d_1\equiv\ldots\equiv d_n+/j$ jt b qpmzopnjbm po $K^{\subsetneq}$0 Y f qspwf cz joevdujpo po n0 Ju jt fbtz up di fdl ui bu jg d jt b efsjwbujpo- ui fo d/j jt beejujwf po $K^{\subsetneq}$0 Tjodf fwfsz beejujwf gvodujpo jt b qpmzopnjbm- ui f tubufnfou jt usvf gps $n\ ?\ 20$

Tvqqptf ui bu $n>2$- boe ui f tubufnfou jt usvf gps $n\quad 20$ Mfu $d_1,\ldots,d_n$ cf efsjwbujpot po K0 Cz ui f joevdujpo i zqpui ftjt- $)d_2\equiv\ldots\equiv d_n+/j\ ?\ p$ jt b qpmzopnjbm po $K^{\subsetneq}$0 Fyufoe p up K cz qvujoh $p)1+?\ 10$ Y f i bwf up ti px ui bu $)d_1\equiv)p\times j++/j$ jt b qpmzopnjbm po $K^{\subsetneq}$0 Tjodf d_1 jt b efsjwbujpo- x f i bwf

$$d_1)p)x+\times x+?\ d_1)p)x++\times x\ ,\ \ p)x+\times d_1)x+$$

gps fwfsz $x\ /\ K$0 Ui vt $)d_1\equiv)p\times j++/j\ ?\ \ )d_1\equiv p+,\ \ p\times)d_1/j+$ po $K^{\subsetneq}$0 Tjodf p jt b qpmzopnjbm boe d_1/j jt beejujwf po $K^{\subsetneq}$- ju gpmmpx t ui bu $p\times)d_1/j+$ jt b qpmzopnjbm po $K^{\subsetneq}$0 Ui fsfgpsf- ju jt fopvhi up ti px ui bu $d_1\equiv p$ jt b qpmzopnjbm po $K^{\subsetneq}$0

Fyufoe d_1 up $\mathbb{C}$ bt b efsjwbujpo0 Mfu $d_1/j\ ?\ a$- ui fo a jt beejujwf po $\mathbb{C}^{\subsetneq}$- boe $d_1\ ?\ a\times j$- x i fsf x f fyufoefe a up $\mathbb{C}$ cz qvujoh $a)1+?\ 10$)Ui f beejujwjuz pg a po $\mathbb{C}^{\subsetneq}$ nfbot

$a)xy+? \ a)x+; \ a)y+$gps fwfsz x, y / $\mathbb{C}^{\subseteq}$0+ Opx p jt b tvn pg gvodujpot pg ui f gpsn $a_1 \times\times\times a_k$- xi fsf fbdi pg $a_1, \ldots, a_k$ jt fjui fs beejujwf po $K^{\subseteq}$ ps dpotubou0 Tjodf d_1 jt beejujwf po $\mathbb{C}$- ju jt fopvhi up ti px ui bu $d_1 \equiv)a_1 \times\times\times a_k+$jt b qpmzopnjbmpo $K^{\subseteq}$0 Yf i bwf

$$\begin{aligned} d_1 \equiv)a_1 \times\times\times a_k+? \ )a \times j+\equiv)a_1 \times\times\times a_k+? \ )a \equiv)a_1 \times\times\times a_k++ \times a_1 \times\times\times a_k \ ? \\ ? \ ])a \equiv a_1+, \ \ldots, \ )a \equiv a_k+ \times a_1 \times\times\times a_k \end{aligned} \qquad)62+$$

fwfszxi fsf po K0 Tjodf $a \equiv a_i$ jt fjui fs dpotubou ps beejujwf po $K^{\subseteq}$- ju gpmmpxt ui bu ui f sjhi u i boe tjef pg)62+jt b qpmzopnjbmpo $K^{\subseteq}$0 □

Pvs ofyu bjn jt up qspwf ui f gpmmpxjoh sftvmu0

Theorem 5.4. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite, and let the map D ; $K \Leftarrow \mathbb{C}$ be additive. Then the following are equivalent.*

)j+ *D is a differential operator on K.*

)jj+ *D/j is a polynomial on $K^{\subseteq}$.*

)jjj+ *D/j is a generalized polynomial on $K^{\subseteq}$.*

)jw+ *D/j is a local polynomial on $K^{\subseteq}$.*

Yf sfnbsl ui bu ui f upstjpo gsff sbol pg ui f Bcfmjbo hspvq $K^{\subseteq}$ jt joffojuf gps boz K0 Joeffe- ui f tfu pg sbujpobm qpxfst pg dpotujuvfout bo joefqfoefou gbn jmz pg fmfn fout pg joffojuf psefs jo $K^{\subseteq}$0 Ui fsfgpsf- gps boz ffmme $K \to \mathbb{C}$- ui f gbn jmjft pg qpmzopnjbmt- hfofsbmj{fe qpmzopnjbmt- boe mpdbmqpmzopnjbmt efffofe po $K^{\subseteq}$ bsf ejtujodu0

Jo ui f ofyu uxp mfnn bt xf ti bmmvtf ui f gpmmpxjoh opubujpo0 Mfu K cf b ffmme bt jo Ui fpsfn 605- boe mfu $T \to K$ cf b n byjn bmtfu pg bmhfcsbjdbmmz joefqfoefou fmfn fout pwfs $\mathbb{Q}$0 Cz bttvn qujpo- T jt ffojuf=mfu $T \ ? \ \}t_1, \ldots, t_n|$ 0 Yf ti bmmefopuf cz G ui f tvchspvq pg $K^{\subseteq}$ hfofsbufe cz $t_1, \ldots, t_n$0 Ui fo G jt b ffojufmz hfofsbufe tvchspvq pg $K^{\subseteq}$0

Lemma 5.5. *Let f ; $K \Leftarrow \mathbb{C}$ be an additive function with respect to addition. Let H be a subgroup of $K^{\subseteq}$ such that $G \to H \to K^{\subseteq}$. Suppose that p ? f/j is a generalized polynomial on H. If $f \subseteq 1$ on G, then $f \subseteq 1$ on H.*

Proof. Yf qspwf cz joevdujpo po ui f efhsff pg ui f hfofsbmj{fe qpmzopnjbmp po H0 Jg efh p ? 1- ui fo p jt dpotubou0 Tjodf $f \subseteq 1$ po G- xf i bwf $p \subseteq 1$ po H- boe $f \subseteq 1$ po H0

Tvqqptf m ? efh $p > 1$- boe ui bu ui f tubufn fou jt usvf gps efhsfft mftt ui bo m0 Mfu g / G cf ffyfe0 Ui fo

$$\begin{aligned} \Lambda_g p)x+? \ p)gx+ \ \ p)x+? \ \frac{f)gx+}{gx} \ \ \frac{f)x+}{x}? \\ ? \ \frac{g \ ^1 f)gx+ \ \ f)x+}{x}? \ \frac{f_1)x+}{x}, \end{aligned} \qquad)63+$$

xifsf $f_1)x+?\ g^{\ 1}f)gx+\ \ f)x+$ gps fwfsz $x\ /\ K$0 Uifo f_1 jt beejujwf po K- boe f_1/j jt b hfofsbmj{fe qpmzopnjbm po H cz)63+0 Npsfpwfs- xf ibwf $f_1/j\ ?\ \Lambda_g p$- boe uivt efh))$f_1/j+\|_H+\geq m$ 20 Tjodf $f_1 \subseteq 1$ po G- ju gpmmpxt gspn uif joevdujpo izqpuiftjt uibu $f_1 \subseteq 1$ po H0 Uivt $f)gx+?\ g\times f)x+$ gps fwfsz $g\ /\ G$ boe $x\ /\ H$0 Cz uif beejujwjuz pg f xf pcubjo

$$f)cx+?\ c\times f)x+\qquad)c\ /\ \mathbb{Q}]T‘,\ x\ /\ H+ \qquad)64+$$

Mfu $\alpha\ /\ H$ cf bscjusbsz0 Uifo- cz $\alpha\ /\ K$- α jt bmhfcsbjd pwfs uif fme $\mathbb{Q})T+$0 Mfu $c_0,\dots,c_k\ /\ \mathbb{Q}]T‘$ cf tvdi uibu

$$c_k\alpha^k,\ \dots,\ c_1\alpha,\ c_0\ ?\ 1, \qquad)65+$$

c_k i? 1 boe k jt njojnbm0 Mfu $f)\alpha^i+?\ a_i$ $)i\ /\ \mathbb{Z}+$0 Nvmujqmzjoh)65+ cz $\alpha^{n\ \ k}$ gps fwfsz $n\ /\ \mathbb{Z}$ xf pcubjo

$$c_k\alpha^n,\ \dots,\ c_1\alpha^{n\ \ k+1},\ c_0\alpha^{n\ \ k}\ ?\ 1.$$

Cz)64+ boe cz uif beejujwjuz pg f- uijt jnqmjft

$$c_k a_n,\ \dots,\ c_1 a_{n\ \ k+1},\ c_0 a_{n\ \ k}\ ?\ 1$$

gps fwfsz n0 Uifsfgpsf- uif tfrvfodf $)a_n+$ tbujtgjft b mjofbs sfdvssfodf sfmbujpo0 Ju jt xfmm lopxo uibu a_n dbo cf vojrvfmz sfqsftfoufe jo uif gpsn $a_n\ ?\ \prod_{\lambda\emptyset\Lambda} q_\lambda)n+\times\lambda^n$- xifsf λ svot uispvhi Σ- uif tfu pg spput pg uif dibsbdufsjtujd qpmzopnjbm $\chi)x+?\ c_kx^k,\ \dots,\ c_0$- boe gps fwfsz sppu $\lambda\ /\ \Sigma$- $q_\lambda\ /\ \mathbb{C}]x‘$ jt b qpmzopnjbm pg uif efhsff mftt uibo uif nvmujqmjdjuz pg λ0

Cz uif njojnbmjuz pg k, uif qpmzopnjbm χ jt jssfevdjcmf pwfs $\mathbb{Q})T+$0 Uifsfgpsf- fwfsz λ jt b tjnqmf sppu pg χ- boe uivt

$$a_n\ ?\ \prod_{\lambda\emptyset\Lambda} d_\lambda \times\lambda^n \qquad)66+$$

gps fwfsz n- xifsf d_λ jt b dpotubou gps fwfsz $\lambda\ /\ \Sigma$0

Tjodf p jt b hfofsbmj{fe qpmzopnjbm po H boe $\}\alpha^n|$ jt b gjojufmz hfofsbufe tvchspvq pg H- ju gpmmpxt uibu p jt b qpmzopnjbm po $\}\alpha^n|$)tff Uifpsfn 3021+0 Uifsfgpsf- uif nbq $n \not\Leftarrow p)\alpha^n+)n\ /\ \mathbb{Z}+$ jt b qpmzopnjbm po $\mathbb{Z}$0 Opx- xf ibwf $a_n\ ?\ f)\alpha^n+?\ p)\alpha^n+\times\alpha^n$ gps fwfsz n0 Uif vojrvfoftt pg uif sfqsftfoubujpo)66+ jnqmjft uibu $\alpha\ /\ \Sigma$ boe uif gvodujpo $n \not\Leftarrow p)\alpha^n+)n\ /\ \mathbb{Z}+$ jt dpotubou0 Tjodf $p)2+?\ f)2+?\ 1$ cz $2\ /\ G$- ju gpmmpxt uibu $p)\alpha^n+?\ 1$ gps fwfsz n0 Jo qbsujdvmbs- $p)\alpha+?\ 1$ boe $f)\alpha+?\ 1$. Tjodf uijt jt usvf gps fwfsz $\alpha\ /\ H$- xf pcubjo $f \subseteq 1$ po H0 □

Lemma 5.6. *Let* $f\ ;\ K \Leftarrow \mathbb{C}$ *be an additive function with respect to addition such that* $p\ ?\ f/j$ *is a local polynomial on* $K^{\subseteq}$. *If* $f \subseteq 1$ *on* G, *then* $f \subseteq 1$ *on* K.

Proof. Ui f beejujwjuz pg f jn qmjft $f)1+?$ 10 Mfu α / $K^{\subseteq}$ cf bscjusbsz- boe mfu H cf ui f n vmujqmjdbujwf hspvq hfofsbufe cz T ? $\}t_1,\ldots,t_n|$ boe α0 Tjodf H jt b ffojufm hfofsbufe tvchspvq pg $K^{\subseteq}$- ju gpmmpxt ui bu p jt b qpmzopn jbmpo H0 Cz ui f qsfwjpvt mfn n b xf pcubjo ui bu $f \subseteq 1$ po H0 Jo qbsujdvmbs- $f)\alpha+?$ 10 Tjodf ui jt jt usvf gps fwfsz α / $K^{\subseteq}$- xf pcubjo $f \subseteq 1$ po $K^{\subseteq}$0 □

Proof of Theorem 5.4. Ui f jn qmjdbujpo)j+$?\Leftrightarrow$)jj+ xbt qspwfe jo Ui fpsfn 604)gps fwfsz fffm+0 Ui f jn qmjdbujpot)jj+$?\Leftrightarrow$)jjj+$?\Leftrightarrow$)jw+ bsf pcwjpvt0

Opx xf qspwf)jw+$?\Leftrightarrow$)j+0 Mfu p ? D/j- ui fo p jt b mpdbmqpmzopn jbmpo $K^{\subseteq}$0 Tjodf G jt b ffojufm hfofsbufe tvchspvq pg $K^{\subseteq}$- ju gpmmpxt cz Qspqptjujpo 3024 ui bu p jt b hfofsbmj{fe qpmzopn jbmboe bmtp b qpmzopn jbmpo G0 Cz Ui fpsfn 305- p i bt ui f P_{loc} qspqfsuz po G= ui bu jt- ui f n bq

$$)k_1,\ldots,k_n+ \forall= p \quad t_1^{k_1} \times\!\times\!\times t_n^{k_n}\Big($$

jt b qpmzopn jbmpo $\mathbb{Z}^n$0

Yf ti bmmvtf ui f opubujpo $x^{[0]}$? 2 boe $x^{[i]}$? $x)x \quad 2+\times\!\times\!\times)x \quad i,\ 2+$ gps fwfsz i ? $2,3,\ldots$ boe x / $\mathbb{Z}$. Jujt xfmml opxo ui bu fwfsz qpmzopn jbmcf mpohjoh up $\mathbb{C}]x_1,\ldots,x_n$‘ dbo cf xsjuufo jo ui f gpsn $\prod c_i \times x_1^{[i_1]} \times\!\times\!\times x_n^{[i_n]}$- xi fsf i ? $)i_1,\ldots,i_n+$ svot ui spvhi b ffojuf tfu pg n.uvqmft pg opoofhbujwf joufhfst- boe jo fbdi ufsn ui f dpfı djfou c_i jt b dpn qmfy ovn cfs0 Ui fsfgpsf- ui f n bq $)k_1,\ldots,k_n+ \forall= p \quad t_1^{k_1} \times\!\times\!\times t_n^{k_n}\Big($ i bt tvdi b sfqsftfoubujpo0 Ui fo xf i bwf

$$\begin{aligned}
D \quad t_1^{k_1} \times\!\times\!\times t_n^{k_n}\Big(&? \; p \quad t_1^{k_1} \times\!\times\!\times t_n^{k_n}\Big(\times t_1^{k_1} \times\!\times\!\times t_n^{k_n} \; ? \\
&? \; \prod c_i \times k_1^{[i_1]} \times\!\times\!\times k_n^{[i_n]} \times t_1^{k_1} \times\!\times\!\times t_n^{k_n} \; ? \\
&? \; \prod c_i \times t_1^{i_1} \times\!\times\!\times t_n^{i_n} \times k_1^{[i_1]} \times\!\times\!\times k_n^{[i_n]} \times t_1^{k_1 \quad i_1} \times\!\times\!\times t_n^{k_n \quad i_n} \; ? \\
&? \; E \quad t_1^{k_1} \times\!\times\!\times t_n^{k_n}\Big(
\end{aligned} \qquad)67+$$

gps fwfsz $k_1,\ldots,k_n$ / $\mathbb{Z}$- xi fsf E jt ui f ejfffsfoujbmpqfsbups

$$\prod c_i \times t_1^{i_1} \times\!\times\!\times t_n^{i_n} \times \frac{\partial^{i_1+\ast\ast\ast+i_n}}{\partial t_1^{i_1} \times\!\times\!\times \partial t_n^{i_n}}.$$

Cz fyufoejoh ui f efsjwbujpot $\partial/\partial t_i$ up K- xf dbo fyufoe E up K bt b ejfffsfoujbmpqfsbups $\overline{E}$0 Ui fo $\overline{E}$ jt beejujwf po K- boe $\overline{E}/j$ jt b qpmzopn jbmpo $K^{\subseteq}$ cz Ui fpsfn 6040 Mfu $q)1+?$ 1- boe mfu $q)x+?$ $p)x+ \quad \overline{E})x+/x$ gps fwfsz x / $K^{\subseteq}$0 Ui fo $q \times j$? $D \quad \overline{E}$ jt beejujwf po K- boe q jt b mpdbmqpmzopn jbmpo $K^{\subseteq}$0 Tjodf q wbojti ft po G cz)67+ ju gpmmpxt gspn Mfn n b 607 ui bu $q \subseteq 1$ po K0 Ui vt D ? $\overline{E}$ po K xi jdi dpn qmfuft ui f qsppg0 □

5.2 Proof of spectral synthesis in varieties of additive functions

Mfu K cf b tvcfffm pg $\mathbb{C}$- boe mfu $K^{\subseteq}$ efopuf ui f Bcfmjbo hspvq }x / K ; x í? 1| xjui sftqf du up n vmjqmjdbujpo0

Jo ui jt tfdujpo xf ffjy b fffm K tvdi ui bu jut usbotdfoefodf efhsff pwfs $\mathbb{Q}$ jt ffojuf0 Yf sfdbmmi bu V_1 efopuft ui f tfu pg beejujwf gvodujpot f ; $K \Leftarrow \mathbb{C}$ boe $V_1^{\subseteq}$? }$f\backslash_{K^*}$; f / V_1| 0 Jg ui f gvodujpo frvbujpo jt usjwjbmmui fo Mfn n b 4033 ti px t ui bu $V_1^{\subseteq}$ jt b wbsjfuz po $K^{\subseteq}$0

Pvs bjn jt up qspwf ui bu tqf dusbmtzoui ftjt i pmt jo fwfsz wbsjfuz po $K^{\subseteq}$ dpoubjofe cz V_1)tff Ui fpsfn 508+0 Ui fpsfn 30 ti px t ui bu bt ui f upstjpo gsff sbol pg $K^{\subseteq}$ jt joffojuf)tff ui f sfn bsl bgufs Ui fpsfn 605+ ui fsf bsf wbsjfujft po $K^{\subseteq}$ jo x i jdi tqf dusbmtzoui ftjt epft opu i pme0

Theorem 5.7. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Let f ; $K \Leftarrow \mathbb{C}$ be additive, and let m be an exponential on $K^{\subseteq}$. Let ϕ be an extension of m to $\mathbb{C}$ as an automorphism of $\mathbb{C}$. Then the following are equivalent.*

)j+ f ? $p \times m$ *on* $K^{\subseteq}$*, where p is a local polynomial on* $K^{\subseteq}$*.*

)jj+ f ? $p \times m$ *on* $K^{\subseteq}$*, where p is a generalized polynomial on* $K^{\subseteq}$*.*

)jjj+ f ? $p \times m$ *on* $K^{\subseteq}$*, where p is a polynomial on* $K^{\subseteq}$*.*

)jw+ *There exists a unique differential operator D on K such that* f ? $\phi \equiv D$ *on K.*

Proof. Ui f jn qmjdbujpot)jjj+? $\Leftrightarrow$)jj+? $\Leftrightarrow$)j+bsf pcwjpvt0

)j+? $\Leftrightarrow$)jw+; Mfu D ? ϕ $^1 \equiv f$0 Ui fo D jt beejujwf po K- boe

$$D\ ?\ \phi\ ^1 \equiv)p \times m+?\ \ )\phi\ ^1 \equiv p+\times j$$

po K0 Ui vt D/j ? ϕ $^1 \equiv p$ po $K^{\subseteq}$0 Tjodf p jt b mpdbmqpmzopn jbmpo $K^{\subseteq}$ boe ϕ ^{1} jt bo bvupn psqi jtn pg $\mathbb{C}$- ju gpmmpx t ui bu ϕ $^1 \equiv p$? D/j jt b mpdbmqpmzopn jbmpo $K^{\subseteq}$ bt x fmm0 Ui fsfgpsf- D jt b ejiffsfoujbmpqfsbups po K cz Ui fpsfn 6050 Tjodf f ? $\phi \equiv D$ po K- ui jt qspwft ui f fyjtufodf pg D0 Ui f vojrvfoftt jt dmfbs; jg $\phi \equiv D_1$? $\phi \equiv D_2$ po K- ui fo D_1 ? D_2- tjodf ϕ jt jokfdujwf0

)jw+? $\Leftrightarrow$)jjj+; Tvqqptf f ? $\phi \equiv D$ po K- x i fsf D jt b ejiffsfoujbmpqfsbups po K0 Ui fo D/j jt b qpmzopn jbmpo $K^{\subseteq}$- boe tp jt

$$p\ ?\ \phi \equiv)D/j+?\ \ )\phi \equiv D+/\phi.$$

Ui vt f ? $\phi \equiv D$? $p \times \phi$. Tjodf m ? ϕ po K- ui f qsppg jt dpn qmfuf0 □

Theorem 5.8. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Then spectral synthesis holds in every variety on $K^{\subseteq}$ consisting of additive functions (with respect to addition).*

Proof. Tjodf K jt dpvoubcmf- tp jt uif Bcfmjbo hspvq $K^\subsetneq$0 Mfu $V \to V_1^\subsetneq$ cf b wbsjfuz po $K^\subsetneq$ dpotjtujoh pg beejujwf gvodujpot0 Cz Uifpsfn 3033- mpdbmtqfdusbmtzouiftjt ipmet po $K^\subsetneq$- boe uivt V jt tqboofe cz mpdbmqpmzopnjbmfyqpofoujbmgvodujpot0 Tjodf- cz Uifpsfn 608- fwfsz mpdbmqpmzopnjbmfyqpofoujbmgvodujpo dpoubjofe cz V jt b qpmzopnjbmfyqpofoujbm gvodujpo- ju gpmmpxt uibu V jt tqboofe cz qpmzopnjbmfyqpofoujbmgvodujpot0 $\square$

5.3 The space of additive solutions

Bt bo bqqmjdbujpo pg Uifpsfnt 608 boe 609 xf eftdsjcf uif beejujwf tpmvujpot pg uif mjofbs gvodujpobmfrvbujpo

$$\prod_{i=1}^{n} a_i f)b_i x\,, \quad c_i y+? \;\; 1, \qquad)68+$$

xifsf a_i, b_i, c_i bsf hjwfo dpnqmfy ovncfst boe f ; $\mathbb{C} \Leftarrow \mathbb{C}$ jt uif volopxo gvodujpo0 Mfu K ? $\mathbb{Q})b_1, \ldots, b_n, c_1, \ldots, c_n$+0 Yf sfdbmmuibu S_1 efopuft uif tfu pg beejujwf tpmvujpot pg)68+efgjofe po K0 Yf tipxfe uibu f ; $K \Leftarrow \mathbb{C}$ cfmpoht up S_1 jg boe pomz jg

$$\prod_{i=1}^{n} a_i f)b_i x+? \;\; 1, \qquad \prod_{i=1}^{n} a_i f)c_i x+? \;\; 1 \qquad)69+$$

ipmet gps fwfsz x / K0 Cz Mfnnb 403-

$$S_1^\subsetneq ? \;\; \}f\backslash_{K^*} \,;\, f \;/\; S_1|$$

jt b wbsjfuz po $K^\subsetneq$0

Uif ofyu uifpsfn jt pvs nbjo sftvmu dpodfsojoh uif beejujwf tpmvujpot pg mjofbs gvod. ujpobmfrvbujpot0

Theorem 5.9.)j+ *For every function f / $S_1^\subsetneq$, f is an exponential monomial on $K^\subsetneq$ if and only if f ? $\phi \equiv D$ on $K^\subsetneq$, where ϕ is an automorphism of $\mathbb{C}$ and is a solution of*)68+*, and D is a differential operator on K.*

)jj+ *The variety $S_1^\subsetneq$ is spanned by the functions*)$\phi \equiv D$+$\backslash_{K^*}$ / $S_1^\subsetneq$*, where ϕ and D are as above.*

)jjj+ *The linear space S_1 is spanned by the functions $\phi \equiv D$, where ϕ and D are as above.*

Proof.)j+Tvqqptf uibu f ; $K \Leftarrow \mathbb{C}$ jt bo beejujwf tpmvujpo pg)68+ boe f jt bo fyqpofoujbm npopnjbmgvodujpo po $K^\subsetneq$0 Mfu f ? $p \times m$- xifsf p jt b qpmzopnjbmboe m jt bo fyqpofoujbm po $K^\subsetneq$0 Tjodf $S_1^\subsetneq$ jt b wbsjfuz boe $p \times m$ / $S_1^\subsetneq$- ju gpmmpxt uibu m / $S_1^\subsetneq$)tff Mfnnb 3026+0 Uijt nfbot uibu efgjojoh m)1+? 1- uif gvodujpo m jt b tpmvujpo pg)68+po K0 Cz Mfnnb

406- m dbo cf fyufoefe bt bo bvupn psqi jtn pg ℂ0 Mfu ϕ efopuf tvdi bo fyufotjpo0 Bt m jt b tpmvujpo pg)69+bt xfmmxf i bwf

$$\prod_{i=1}^{n} a_i m)b_i+? \ 1, \qquad \prod_{i=1}^{n} a_i m)c_i+? \ 1. \qquad)6:+$$

Ui fo- cz)6: + ϕ jt b tpmvujpo pg)68+po ℂ0 Ui f sftu pg ui f tubufn fou)j+gpmmpx t gspn Ui fpsfn 6080

Tubufn fou)jj+jt bo jn n fejbuf dpotfrvfodf pg Ui fpsfn 609 boe pg)j+0 Ui f tubufn fou)jjj+jt dmfbs gspn)jj+0 □

Ui f eftdsjqujpo pg ui f beejujwf tpmvujpot cfdpn ft ftqfdjbmmz tjn qmf jg ui f dpfı djfout a_i bsf bmhfcsbjd0

Theorem 5.10. *Suppose that $a_1,\dots,a_n$ are algebraic numbers. If ϕ is an automorphism of ℂ and ϕ is a solution of)68+, then $\phi\equiv D$ / S_1 for every differential operator D on K. Therefore, $S_1^{\subsetneq}$ is spanned by the functions)$\phi\equiv D$+$|_{K^*}$ / $S_1^{\subsetneq}$, where ϕ is an automorphism of ℂ and is a solution of)68+, and D is an arbitrary differential operator on K.*

Proof. Tjodf xf bsf pomz jouf sftufe jo ui f beejujwf tpmvujpot pg)68+ ju jt fopvhi up efbm xjui ui f beejujwf tpmvujpot pgui f tztufn)69+0 Ju jt fopvhi up ti px ui bu $\phi\equiv D$ jt b tpmvujpo pg)69+po K gps boz ejfifsfoujbmpqfsbups D ? $d_1\equiv\times\times\times\equiv d_k$- xi fsf $d_1,\dots,d_k$ bsf efsjwbujpot0 Yf xjmmqspwf ui jt cz joevdujpo po k0

Jg k ? 1=ui bu jt- jg D jt ui f jefoujuz- ui fo $\phi\equiv D$? ϕ jt b tpmvujpo cz bttvn qujpo0

Mfu $k > 1$- boe tvqqptf ui f tubufn fou jt usvf gps k 20 Yf i bwf up qspwf ui bu jg $d_1,\dots,d_k$ bsf efsjwbujpot po K- ui fo $\phi\equiv)d_1\equiv\times\times\times\equiv d_k$+jt b tpmvujpo po K0 Yf i bwf

$$\phi\equiv)d_1\equiv\times\times\times\equiv d_k+? \ d\equiv f,$$

xi fsf d ? $\phi\equiv d_1\equiv\phi$ ^{1} boe f ? $\phi\equiv)d_2\equiv\times\times\times\equiv d_k$+0 Ui fo f ; $K\Leftarrow\phi)K$+jt b tpmvujpo pg)69+ cz ui f joevdujpo i zqpui ftjt0

Mfu K_1 ? $K)\phi$ $^1)a_1+,\dots,\phi$ $^1)a_n$++ boe mfu d_1 cf fyufoefe up K_1 bt b efsjwbujpo0 Yf efopuf ui f fyufoefe efsjwbujpo cz $\overline{d}_1$0 Opuf ui bu $\overline{d}_1)a+?$ 1 gps fwfsz bmhfcsbjd fmfn fou pg $K_1$0

Mfu $\overline{d}$? $\phi\equiv\overline{d}_1\equiv\phi$ ^{1}0 Ju jt fbtz up di fdl ui bu $\overline{d}$ jt b efsjwbujpo po $\phi)K_1$+0 Jg a / $\phi)K_1$+ jt bmhfcsbjd- ui fo tp jt ϕ $^1)a$+ boe ui vt $\overline{d}_1)\phi$ $^1)a$++? 10 Ui fsfgpsf- $\overline{d})a$+? 1 gps fwfsz bmhfcsbjd fmfn fou pg $\phi)K_1$+0 Jo qbsujdvmbs- $\overline{d})a_i$+? 1 gps fwfsz i ? $2,\dots,n$0 Tjodf f jt b tpmvujpo pg)69+xf i bwf- gps fwfsz x / K-

$$1 \ ? \ \overline{d})1+? \ \overline{d}\Bigg)\prod_{i=1}^{n} a_i \times f)b_i x+\Bigg(\ ? \ \prod_{i=1}^{n}\overline{d})a_i\times f)b_i x+++?$$

$$? \ \prod_{i=1}^{n}\overline{d})a_i+\times f)b_i x+, \ \prod_{i=1}^{n} a_i \times d)f)b_i x++? \ \prod_{i=1}^{n} a_i \times)d\equiv f+)b_i x+\!+$$

Ui f tbn f bshvn fou ti px t ui bu $\prod_{i=1}^{n} a_i \times)d \equiv f+)c_i x+ ?\ 1$ gps fwfsz x / K0 Ui vt $\phi \equiv d$ jt b tpmvujpo pg)69+po K xi jdi dpn qmfuft ui f qsppg0 □

Cz Ui fpsfn t 60 boe 6021 xf i bwf ui f gpmmpx joh dpspmmbsz0

Corollary 5.11. *If the coefficients $a_1, \ldots, a_n$ are algebraic, then the variety of additive solutions of)68+defined on K is spanned by the functions $\phi \equiv D$, where ϕ is an isomorphism solution and D is an arbitrary differential operator.* □

Theorem 5.12. *Suppose that $a_1, \ldots, a_n$ are algebraic and that)68+has a nonzero additive solution on K. Then S_1 is of finite dimension over $\mathbb{C}$ if and only if each of the numbers $b_1, \ldots, b_n, c_1, \ldots, c_n$ is algebraic.*

Proof. Jg b_i boe c_i bsf bmhfcsbjd ovn cfst- ui fo ui f ffmn

$$K\ ?\ \mathbb{Q})b_1, \ldots, b_n, c_1, \ldots, c_n+$$

jt b ffojuf ejn fotjpobmfyufotjpo pwfs $\mathbb{Q}$)tff ui f qsppg pg Ui fpsfn 504+0 Dpotfrvfoumz- ui f mjofbs tqbdf pg beejujwf gvodujpot efffofe po K jt bmtp pgffojuf ejn fotjpo0 Ui jt jn qmjft ui bu ui f tqbdf pg beejujwf tpmvujpot pg)68+jt- b gpsujpsj- pg ffojuf ejn fotjpo0

Ofyu tvqqptf ui bu bu mfbtu pof pg ui f ovn cfst $b_1, \ldots, b_n, c_1, \ldots, c_n$ jt usbotdfoefoubm0 Yf ti px ui bu jg)68+i bt b opo{fsp beejujwf tpmvujpo po K- ui fo ui f tfu S_1 pg bmmopo{fsp beejujwf tpmvujpo efffofe po K i bt joffojuf ejn fotjpo pwfs $\mathbb{Q}$0

Cz Ui fpsfn 408)ps- cz)jjj+pg Ui fpsfn 60 + ui fsf jt bo bvupn psqi jtn ϕ pg $\mathbb{C}$ xi jdi jt b tpmvujpo pg)68+po K0 Mfu T cf b n byjn bmtvctfu pg K dpotjtujoh pg bmhfcsbjdbmmz joefqfoefoufmfn fout pwfs $\mathbb{Q}$0 Tjodf ui f efhsff pg usbotdfoefodf pg K jt bu mfbtu 2- T ?. A0 Mfu t / T cf tfmfdufe0 Gps fwfsz n ui fsf jt b ejfifsfoujbmpqfsbups D_n po K xi jdi jt bo fyufotjpo pg $\frac{\partial^n}{\partial t^n}$ gspn $\mathbb{Q})T+$0 Dmfbsmz- ui f pqfsbupst D_n bsf mjofbsmz joefqfoefou pwfs $\mathbb{Q})T+$0 Ui fo tp bsf ui f n bqt $\phi \equiv D_n$0 Tjodf- cz Ui fpsfn 6021- ui f n bqt $\phi \equiv D_n$ bsf beejujwf tpmvujpot pg)68+po K- ui f qsppg jt dpn qmfuf0 □

Jg b_i boe c_i bsf bmhfcsbjd ovn cfst- ui fo fwfsz ejfifsfoujbmpqfsbups po K jt b dpotubou n vmujqmf pg ui f jefoujuz0 Ui fsfgpsf- jo ui jt dbtf b ffojuf cbtjt pg ui f mjofbs tqbdf S_1 dpotjtut pg ui f jokfdujwf i pn pn psqi jtn t tbujtgzjoh

$$\prod_{i=1}^{n} a_i \phi_j)b_i+ ?\ 1 \text{ boe } \prod_{i=1}^{n} a_i \phi_j)c_i+ ?\ 1 \qquad)71+$$

gps fwfsz j / $\}2, \ldots, k|$ 0 Yf efbmx jui ui jt dbtf jo Ui fpsfn t 504 boe 505)Tff bmtp]24‘+0

Fybn qmf 509 ti px t ui bu jg ui f ovn cfst b_i boe c_i bsf opu bmmbmhfcsbjd- ui fo ui f jokfdujwf i pn pn psqi jtn tpmvujpot ep opu ofdfttbsjmz tqbo S_1=ui bu jt- xf n bz offe opousjwjbm ejfifsfoujbmpqfsbupst jo psefs up hfofsbuf $S_1$0 Pof dbo ti px ui bu $\partial/\partial t^k$ jt opu b tpmvujpo pg)59+jg $k \subset 3$- boe ui vt S_1 jt pg ffojuf ejn fotjpo pwfs $\mathbb{Q})t+0$ Jo ui f ofyu ui fpsfn xf ti px ui bu ui jt cfi bwjpvs jt uzqjdbmtvqqptjoh ui bu b_i, c_i hfofsbuf b qvsfm usbotdfoefoubmfffme pg usbotdfoefodf efhsff 20

Y f sfn joe ui bu b gvodujpobmfr vbujpo pg ui f gpsn)68+jt dbmfe *trivial* jg f wfsz beejujwf gvodujpo f ; $\mathbb{C} \Leftarrow \mathbb{C}$ jt b tpmvujpo0

Theorem 5.13. *Suppose that $b_1, \ldots, b_n, c_1, \ldots, c_n$ / $\mathbb{Q})t+$, where t is transcendental over $\mathbb{Q}$. Then the equation)68+is either trivial or S_1 is of finite dimension over $\mathbb{C}$.*

Proof. Ui f beejujwf tpmvujpot pg)68+bsf ui f tbn f bt ui f tpmvujpot pg ui f tztufn)69+0 Jg c pui pg ui f fr vbujpot pg)69+bsf usjwjbmui fo tp jt)68+0 Bmtp- jg ui f tqbdf pg beejujwf tpmvujpot pg boz pg ui f fr vbujpot pg)69+jt pg ffojuf ejn fotjpo- ui fo ui f tbn f jt usvf gps ui f tqbdf pg beejujwf tpmvujpot pg)68+0 Ui fsfgpsf- ju jt fopvhi up ti px ui bu jg ui f fr vbujpo $\prod_{i=1}^{n} a_i f)b_i x+?\ 1$ jt opousjwjbmui fo ui f mjofbs tqbdf pg jut beejujwf tpmvujpot efffofe po $\mathbb{Q})t+$jt pg ffojuf ejn fotjpo0

Gps f wfsz γ /? 1- ui f fr vbujpot $\prod_{i=1}^{n} a_i f)b_i x+?\ 1$ boe

$$\prod_{i=1}^{n} a_i f)b_i \gamma x+?\ 1 \qquad)72+$$

bsf fr vjwbmfou jo ui f tfotf ui bu jg pof pg ui f fr vbujpot jt usjwjbmui fo tp jt ui f pui fs- boe jg ui f tqbdf pg beejujwf tpmvujpot pg pof pg ui fn jt pg ffojuf ejn fotjpo- ui fo ui f tbn f jt usvf gps ui f pui fs0

Cz bttvn qujpo- $b_1, \ldots, b_n$ bsf sbujpobmgvodujpot pg t xjui sbujpobmdpfi djfout0 Mfu γ efopuf ui f dpn n po efopn jobups pg $b_1, \ldots, b_n$0 Ui fo $b_i\gamma$ / $\mathbb{Q}]t$'0 Mfu $b_i\gamma\ ?\ \prod_{j=0}^{m} \alpha_{i,j} t^j$- xi fsf $\alpha_{i,j}$ / $\mathbb{Q}$ gps f wfsz i, j0 Tjodf xf bsf joufsftufe jo ui f beejujwf tpmvujpot pom- xf n bz sfqmbdf fbdi ufsn $f)b_i\gamma x+$cz $\prod \alpha_{i,j} f)t^j x+$)i fsf xf vtfe ui f gbdu ui bu $f)\alpha x+?\ \alpha \times f)x+$gps f wfsz sbujpobm$\alpha$+0 Dpmfdujoh ui f ufsn t $a_i \times \alpha_{i,j} f)t^j x+$jo ui f tvn $\prod_{i=1}^{n} a_i f)b_i\gamma x+$gps f wfsz j- xf ffoe ui bu ui fsf jt bo fr vbujpo

$$\prod_{j=0}^{m} A_j f)t^j x+?\ 1 \qquad)73+$$

tvdi ui bu ui f beejujwf tpmvujpot pg)72+boe ui ptf pg)73+dpjodjef0 Jg $A_j\ ?\ 1$ gps f wfsz $j\ ?\ 1, \ldots, m$- ui fo ui f fr vbujpot bsf usjwjbm0 Ui fsfgpsf- xf n bz bttvn f ui bu A_j /? 1 gps tpn f j0 Y f qspwf ui bu jo ui jt dbtf ui f tfu pg beejujwf tpmvujpot pg)73+efffofe po $\mathbb{Q})t+$jt pg ffojuf ejn fotjpo0

Mfu ff efopuf ui f tfu pg gvodujpot $\phi \equiv D$- xi fsf ϕ jt bo bvupn psqi jtn pg $\mathbb{C}$ boe jt b tpmvujpo pg)73+ D jt b ejfifsfoujbmpqfsbups pg $\mathbb{Q})t+$ boe $\phi \equiv D$ jt b tpmvujpo pg)73+0 Cz

Ui f psfn 60 - bqqrjfe up ui f frvbujpo $\prod_{j=0}^{m} A_j f)t^j x$, 1 $\times y$+? 1- x f ffoe ui bu ui f mjofbs tqbdf pg beejujwf tpmvujpot pg)73+jt tqboofe cz ff boe ui vt ju jt fopvhi up ti px ui bu ff hfofsbuft b ffojuf ejn fotjpobm mjofbs tqbdf pwfs $\mathbb{C}$0

Jg ϕ jt bo bvupn psqi jtn pg $\mathbb{C}$ boe b tpmvujpo pg)73+ ui fo x f i bwf

$$\prod_{j=0}^{m} A_j \phi)t^j x+? \ 1$$

gps fwfsz x / $\mathbb{Q}$)t+0 Qvuujoh x ? 2 boe vtjoh ϕ)t^j+?)ϕ)t++^{j} x f ffoe ui bu ϕ)t+jt b sppu pg ui f qpmzopn jbm P)X+? $\prod_{j=0}^{m} A_j \times X^j$0 Tjodf P pomz i bt b ffojuf ovn cfs pg sppput boe ϕ jt efufsn jofe cz ui f wbmvf pg ϕ)t+ x f pcubjo ui bu ui f ovn cfs pg qpttjcmf ϕ(t jt ffojuf0

Dpotfrvfoumz- ju jt fopvhi up ti px ui bu jg ϕ jt ffyfe- ui fo ui ptf ejfifsfoujbm pqfsbupst D gps x i jdi $\phi \equiv D$ / S_1 dpotujuvuf b ffojuf ejn fotjpobm tqbdf pwfs $\mathbb{C}$0

Gjy ϕ- boe mfu D ? $\prod_{k=0}^{s} d_k \frac{\partial^k}{\partial t^k}$ cf b ejfifsfoujbm pqfsbups tvdi ui bu d_s i? 1 boe $\phi \equiv D$ / $S_1$0 Yf qspwf ui bu $s \geq m$0 Tjodf $\phi \equiv D$ jt b tpmvujpo- x f i bwf

$$\prod_{j=0}^{m} A_j \prod_{k=0}^{s} \delta_k \times \phi \Big) \frac{\partial^k}{\partial t^k})t^j x+ \Big(\ ? \ 1$$

gps fwfsz x / $\mathbb{Q}$)t+ x i fsf δ_k ? ϕ)d_k+0 Tjodf

$$\frac{\partial^k}{\partial t^k})t^j x+? \prod_{i=0}^{k} \Big)^k_i \Big(\times \frac{\partial^{k\ i}}{\partial t^{k\ i}})t^j + \times \frac{\partial^i}{\partial t^i} x,$$

x f pcubjo

$$\prod_{i=0}^{s} B_i \times \phi \Big) \frac{\partial^i}{\partial t^i} x \Big(\ ? \ 1, \qquad)74+$$

x i fsf

$$\begin{aligned} B_i \ ? & \prod_{j=0}^{m} \prod_{k=i}^{s} A_j \times \Big)^k_i \Big(\times \delta_k \times \phi \Big) \frac{\partial^{k\ i}}{\partial t^{k\ i}} t^j \Big(\ ? \\ ? & \prod_{j=0}^{m} \prod_{\nu=0}^{s\ i} \Big)^{\nu\ ,\ i}_{\ i} \Big(\times \delta_{\nu+i} \times A_j \times \phi \Big) \frac{\partial^\nu}{\partial t^\nu} t^j \Big(\ ? \qquad)75+ \\ ? & \prod_{\nu=0}^{s\ i} \Big)^{\nu\ ,\ i}_{\ i} \Big(\times \delta_{\nu+i} \times \ _\nu = \end{aligned}$$

i fsf x f vtfe ui f opubujpo

$$_\nu \ ? \prod_{j=0}^{m} A_j \times \phi \Big) \frac{\partial^\nu}{\partial t^\nu} t^j \Big(. \qquad)76+$$

Bqqmzjoh)74+x jui x ? 2 x f pcubjo B_0 ? 10 Ui fo qvuujoh x ? t joup)74+x f pcubjo B_1 ? 10 Yf dpoujovf- cz tvctujuvujoh $t^2, t^3, \ldots$ joup)74+ boe ffoe ui bu B_i ? 1 gps fwfsz i ? $1, \ldots, s$0

Opx uif frvbujpo B_s ? 1 hjwft $_0$? 1 cz d_s î? 10 Uifo- gspn B_{s} $_1$? 1 xf pcubjo $_1$? 10 Dpoujovjoh uijt xbz xf ffioe uibu $_\nu$? 1 gps fwfsz ν ? $1, \dots, s$0

Ju jt fbtz up difdl uibu

$$\phi\Big)\frac{\partial^\nu}{\partial t^\nu}t^j\Big(\; ? \; \Big)\frac{\partial^\nu}{\partial X^\nu}X^j\Big(_{X=\phi(t)}$$

gps fwfsz ν, j ? $1, 2, \dots$0 Uifsfgpsf- cz)76+ $_\nu$? 1 hjwft

$$\begin{aligned} 1 \; ? \; {}_\nu \; ? \; & \prod_{j=0}^{m} A_j \times \phi\Big)\frac{\partial^\nu}{\partial t^\nu}t^j\Big(\; ? \\ ? \; & \prod_{j=0}^{m} A_j \times\Big)\frac{\partial^\nu}{\partial X^\nu}X^j\Big(_{X=\phi(t)} \; ? \\ ? \; & \frac{\partial^\nu}{\partial X^\nu}\Big)\prod_{j=0}^{m} A_j X^j\Big(_{X=\phi(t)} \; ? \\ ? \; & P^{(\nu)})\phi)t++ \end{aligned}$$

Tjodf uijt jt usvf gps fwfsz ν ? $1, \dots, s$- xf pcubjo uibu $\phi)t+$jt b sppu pg P pg nvmujqmjdjuz bu mfbtu s0 I pxfwfs- P jt b opo–fsp qpmzopnjbm pg efhsff bu n ptu m- xijdi hjwft $s \geq m$0

Yf ibwf qspwfe uibu jg D jt b ejggfsfoujbm pqfsbups po $\mathbb{Q})t+$tvdi uibu $\phi \equiv D$ jt b tpmvujpo pg)68+po $\mathbb{Q})t+$ uifo uif efhsff pg D jt bu n ptu m0 Uijt jnqmjft uibu uif tfu pg uiftf gvodujpot $\phi \equiv D$ hfofsbuf b mjofbs tqbdf pg ffojuf ejn fotjpo- xijdi dpn qmfuft uif qsppg0 □

Tvqqptf uibu $a_1, \dots, a_n$ bsf bmhfcsbjd boe b_i, c_i hfofsbuf b qvsfmz usbotdfoefoubm ffme pg usbotdfoefodf efhsff 20 Uifo ju gpmmpxt gspn Uifpsfn t 6023 boe 6024 uibu jg)68+ibt b opu jefoujdbmmz –fsp beejujwf tpmvujpo po K- uifo uif frvbujpo jt usjwjbm0

Uif gpmmpxjoh sftvmu hfofsbmj–ft uijt pctfswbujpo boe ju jt bmtp b hfofsbmj–bujpo pg uif Uifpsfn t 3013)j+boe Uifpsfn 4025)j+0

Theorem 5.14. *Suppose that $a_1, \dots, a_n$ are algebraic and the field K is purely transcendental where K ? $\mathbb{Q})b_1, \dots, b_n, c_1, \dots, c_n+$ If)68+ has a not identically zero additive solution on K, then the equation is trivial.*

Proof. Mfu K ? $\mathbb{Q})t_1, \dots, t_k+$ xifsf $t_1, \dots, t_k$ bsf bmhfcsbjdbmmz joefqfoefou pwfs $\mathbb{Q}$0 Bqqmzjoh uif bshvn fou pg uif qsppg pg Uifpsfn 6024- ju jt fopvhi up qspwf uif gpmmpxjoh tubufn fou0 Dpotjefs uif frvbujpo

$$\prod_{i_1 \dots i_k = 0}^{m} A_{i_1 \dots i_k} \times f\;\; t_1^{i_1} \dots t_k^{i_k} \times x\Big(\; ? \; 1, \qquad)77+$$

xifsf uif dpfi djfout $A_{i_1 \dots i_k}$ bsf bmhfcsbjd0 Jg)77+ibt bo beejujwf tpmvujpo po K xijdi jt opu jefoujdbmmz –fsp- uifo uif frvbujpo jt usjwjbm0

Ju jt fbtz up di fdl ui bu fwfsz beejujwf tpmvujpo pg)77+efffiofe po K dbo cf fyufoefe up $\mathbb{C}$ bt bo beejujwf tpmvujpo po $\mathbb{C}$0 Ui fsfgpsf- jg)77+i bt bo beejujwf tpmvujpo po K x i jdi jt opu jefoujdbmmz –fsp- ui fo ui fsf jt tvdi b tpmvujpo po $\mathbb{C}$0 Cz Ui fpsfn 408)ps- cz)jjj+pg Ui fpsfn 60 + ju gpmmpx t ui bu ui fsf jt bo bvupn psqi jtn ϕ pg $\mathbb{C}$ x i jdi jt b tpmvujpo0 Ui jt n fbot ui bu

$$\prod_{i_1\dots i_k=0}^{m} A_{i_1\dots i_k} \times)\phi)t_1++^{i_1}\dots)\phi)t_k++^{i_k} \ ?\ 1.$$

Tjodf $\phi)t_1+,\dots,\phi)t_k+$ bsf bmhfcsbjdbmmz joefqfoefou pwfs $\mathbb{Q}$ boe ui f ovn cfst $A_{i_1\dots i_k}$ bsf bmhfcsbjd- ju gpmmpx t ui bu fbdi $A_{i_1\dots i_k}$ frvbmt –fsp0 Ui fo ui f frvbujpo)77+jt pcwjpvtmz usjwjbm0 □

6 Spectral synthesis of higher degree

6.1 Proof of spectral synthesis in varieties of k-additive functions

Jo uijt tfdujpo pvs bjn jt up qspwf uif ijhifs efhsff bobmphvf pg Uifpsfn 6080 Yf tibmm offe uif gpmmpxjoh opubujpo0

Jg $\phi_1, \dots, \phi_k$ bsf bvupn psqijtn t pg $\mathbb{C}$- uifo $\mathcal{T}_{(\phi_1,\dots,\phi_k)}$ efopuft uif tfupgui ptf gvodujpot xi jdi bsf ffojuf tvn t pg gvodujpot pg uif gpsn

$$)x_1, \dots, x_k+\not\mapsto \prod_{i=1}^{k})\phi_i \equiv D_i+)x_i+ \qquad)x_1, \dots, x_n \;/\; K+; \qquad)78+$$

xifsf $D_1, \dots, D_k$ bsf ejfifsfoujbmpqfsbupst po K0 Cz $\mathcal{T}_{\mathcal{D}}$ xf n fbo uif dmbtt pg dpotubou gvodujpot0

Yf tibmmefopuf cz $\mathcal{D}_{(\phi_1,\dots,\phi_k)}$ uif tfu pg gvodujpot $\bigcup_{i=1}^{k})\phi_i \equiv D_i+$ efffiofe po K- xifsf $D_1, \dots, D_k$ bsf bt bcpwf0 Ui bu jt-

$$\mathcal{D}_{(\phi_1,\dots,\phi_k)} \;?\; \}\mathrm{ejbh}\, F \;;\; F \;/\; \mathcal{T}_{(\phi_1,\dots,\phi_k)}| \, , \qquad)79+$$

xifsf ejbh F efopuft uif ejbhpobmpg uif gvodujpo $F \;;\; K^k \Leftarrow \mathbb{C}$ efffiofe cz $f)x+ \;?\; F)x, \dots, x+)x \;/\; K+0$

Sfdbmmui bu $K^{\subseteq}$ efopuft uif Bcfmjbo hspvq $\}x \;/\; K \;;\; x \;?\; 1|$ xjui sftqfdu up n vmujqmjdb- ujpo0

Theorem 6.1. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Let $F \;;\; K^k \Leftarrow \mathbb{C}$ be a k-additive function. Let $m_1, \dots, m_k$ be injective homomorphisms of K, let*

$$m)x+ ? \; m)x_1, \dots, x_k+ ? \; m_1)x_1+ \times\times\times m_k)x_k+$$

for every $x_1, \dots, x_k \;/\; K$, and let ϕ_i be an extension of m_i to $\mathbb{C}$ as an automorphism of $\mathbb{C}$ for every $i \;?\; 2, \dots, k$. Then the following are equivalent.

)j+ *$F \;?\; p \times m$ on $)K^{\subseteq}+^k$, where p is a local polynomial on $)K^{\subseteq}+^k$.*

)jj+ *$F \;?\; p \times m$ on $)K^{\subseteq}+^k$, where p is a generalized polynomial on $)K^{\subseteq}+^k$.*

)jjj+ *$F \;?\; p \times m$ on $)K^{\subseteq}+^k$, where p is a polynomial on $)K^{\subseteq}+^k$.*

)jw+ *$F \;/\; \mathcal{T}_{(\phi_1,\dots,\phi_k)}$.*

Proof. Uif jn qmjdbujpot)jjj+? $\Leftrightarrow$)jj+? $\Leftrightarrow$)j+ bsf pcwjpvt0

)j+? $\Leftrightarrow$)jw+; Yf qspwf cz joevdujpo po k0 Uif dbtf $k \;?\; 2$ jt dpwfsfe cz Uifpsfn 6080 Mfu $k > 2$- boe tvqqptf uif tubufn fou jt usvf gps k 20 Jo uif qsppg pg uif joevdujpo tufq xf ti bmbttvn f ui bu k ? 30 Ju jt f btz up di fdl ui bu ui f tbn f bshvn fou x psl t jo ui f hfofsbm dbtf0

Tvqqptf F jt cjbeejujwf po K^2- boe F ? $p \times m$ po)$K^{\subseteq}$+^{2}- x i fsf p jt b mdbmqpmopn jbm po)$K^{\subseteq}$+^{2}- boe

$$m)x, y+? \ m_1)x+\times m_2)y+$$

gps fwfsz x, y / K- x i fsf m_1 boe m_2 bsf hjwfo jolfdujwf i pn pn psqi jtn t pg K0 Mfu ϕ cf bo fyufotjpo pg m_1 boe ψ cf bo fyufotjpo pg m_2 up $\mathbb{C}$ bt bvupn psqi jtn t0

Mfu T cf b n byjn bmtvctfu pg K dpotjtujoh pg bmhfcsbjdbmm joefqfoefou fmfn fout pwfs $\mathbb{Q}$0 Ui fo T jt ffojuf=mfu T ? $\}t_1, \ldots, t_N|$ 0 Mfu G efopuf ui f tvchspvq pg $K^{\subseteq}$ hfofsbufe cz T0 Ui fo G^2 jt b ffojufm hfofsbufe tvchspvq pg)$K^{\subseteq}$+^{2}- boe ui vt p jt b qpmopn jbmpo $G^2$0 Ui fsfgpsf- p jt b ffojuf tvn pg ufsn t pg ui f gpsn $a_1 \times\times\times a_s$- x i fsf fbdi gbdups a_i jt fjui fs beejujwf po)$K^{\subseteq}$+^{2} ps jt b dpotubou0 Opuf ui bu ui f beejujwjuz pg ui f gvodujpo a ;)$K^{\subseteq}$+^{2} $\Leftarrow \mathbb{C}$ n fbot

$$a)uv+? \ a)u+, \ a)v+ \qquad)u, v \ / \ )K^{\subseteq}+^2+.$$

Mfu x ? $t_1^{j_1} \times\times\times t_N^{j_N}$- y ? $t_1^{k_1} \times\times\times t_N^{k_N}$ cf bscjusbsz fmfn fout pg G- x i fsf $j_1, \ldots, j_N, k_1, \ldots, k_N$ / $\mathbb{Z}$0 Mfu a ;)$K^{\subseteq}$+^{2} $\Leftarrow \mathbb{C}$ cf beejujwf0 Jg $a)t_i, 2+? \ \alpha_i$ boe $a)2, t_i+? \ \beta_i$- ui fo

$$a)x, y+? \ \alpha_1 j_1 \ , \ \ldots, \ \alpha_N j_N \ , \ \beta_1 k_1 \ , \ \ldots, \ \beta_N k_N$$

boe

$$m)x, y+? \ \phi)t_1+^{j_1} \times\times\times\phi)t_N+^{j_N} \times\psi)t_1+^{k_1} \times\times\times\psi)t_N+^{k_N}$$

gps fwfsz $j_1, \ldots, j_N, k_1, \ldots, k_N$ / $\mathbb{Z}$0 Ui fsfgpsf- ui f wbmvf pg ui f gvodujpo $a_1 \times\times\times a_s \times m$ bu ui f qpjou)x, y+jt b mjofbs dpn cjobujpo- x ji dpn qmfy dpf ı djfout- pg ufsn t pg ui f gpsn

$$\begin{aligned} & j_1^{c_1} \times\times\times j_N^{c_N} \times k_1^{d_1} \times\times\times k_N^{d_N} \times\phi)t_1+^{j_1} \times\times\times\phi)t_N+^{j_N} \times\psi)t_1+^{k_1} \times\times\times\psi)t_N+^{k_N} \ ? \\ & ? \ \phi \ j_1^{c_1} \times\times\times j_N^{c_N} \times t_1^{j_1} \times\times\times t_N^{j_N} \Big(\times\psi \Big) k_1^{d_1} \times\times\times k_N^{d_N} \times t_1^{k_1} \times\times\times t_N^{k_N} \Big(, \end{aligned} \qquad)7{:}+$$

x i fsf c_i, d_i bsf opoofhbujwf joufhfst0 Ui fo ui f wbmvf pg $p \times m$ bu ui f qpjou)x, y+jt bmtp b mjofbs dpn cjobujpo pg ufsn t pg ui f tbn f gpsn 0

Ju jt f btz up tff ui bu gps fwfsz di pjdf pg ui f opoofhbujwf joufhfst c_i, d_i ui fsf bsf ejfifs. foujbmpqfsbupst D boe E po $\mathbb{Q})T$+tvdi ui bu

$$D \ t_1^{j_1} \times\times\times t_N^{j_N} \Big(? \ j_1^{c_1} \times\times\times j_N^{c_N} \times t_1^{j_1} \times\times\times t_N^{j_N},$$

boe

$$E \Big) t_1^{k_1} \times\times\times t_N^{k_N} \Big(? \ k_1^{d_1} \times\times\times k_N^{d_N} \times t_1^{k_1} \times\times\times t_N^{k_N}$$

gps fwfsz $j_1, \ldots, j_N, k_1, \ldots, k_N$ / $\mathbb{Z}$)tff ui f qsppg pg Ui fpsfn 605+0 Opx ju gpmpx t gspn)7: +ui bu ui f n bq $p \times m$- sftusjdufe up G^2- jt ui f ffojuf tvn pg ui f gpsn

$$)\phi \equiv D+)x+\times)\psi \equiv E+)y+ \qquad)x, y \ / \ G+,$$

xifsf D boe E bsf ejfifsfoujbmpqfsbupst0 Mfu

$$)p \times m+)x,y+? \prod_{\nu=1}^{S})\phi \equiv D_\nu+)x+\times)\psi \equiv E_\nu+)y+ \qquad)81+$$

gps fwfsz x,y / G0 Uif n bqt D_ν, E_ν dbo cf fyufoefe up K bt ejfifsfoujbmpqfsbupst0 Uifo uif fyufoefe n bqt)efopufe cz uif tbn f mfuufs+bsf beejujwf po K boe $D_\nu/j, E_\nu/j$ bsf qpmzopn jbmt po $K^{\subseteq}$0 Uif fyufoefe ejfifsfoujbmpqfsbupst n blf uif sjhiu iboe tjef pg)81+ xfmmefgjofe po)$K^{\subseteq}$+^{2}0 Yf qspwf uibu)81+ipmet fwfszxifsf po)$K^{\subseteq}$+^{2}0

Mfu x / G cf gjyfe0 Uifo uif mfgu iboe tjef pg)81+frvbmt $q)y+\times m_2)y+$ xifsf $q)y+? p)x,y+\times m_1)x+$ gps fwfsz y / K0 Ju jt fbtz up difdl uibu uif gvodujpo $y \not\leftarrow p)x,y+$ jt b mpdbm qpmzopn jbmpo $K^{\subseteq}$ boe uivt tp jt q0

Mfu)$\phi \equiv D_\nu$+)x+? γ_ν boe ψ^{1})γ_ν+? δ_ν0 Uifo uif sjhiu iboe tjef pg)81+frvbmt $\psi \equiv E$- xifsf $E ? \prod_{\nu=1}^{S} \delta_\nu E_\nu$ jt b ejfifsfoujbmpqfsbups po K0 Cz)81+

$$q)y+\times m_2)y+? \ )\psi \equiv E+)y+$$

po G0 Cz uif frvjwbmfodf pg uif tubufn fout)j+boe)jw+pg Uifpsfn 608- uifsf jt b vojrvf ejfifsfoujbmpqfsbups $\overline{E}$ po K tvdi uibu

$$q)y+\times m_2)y+? \ )\psi \equiv \overline{E}+)y+$$

gps fwfsz y / K0 Uifo $\psi \equiv E ? \psi \equiv \overline{E}$ po G- tjodf cpui tjeft frvbm $q \times m_2$ po G0 Tjodf ψ jt jokfdujwf- uijt jn qmjft $E ? \overline{E}$ po G0

Tjodf E boe $\overline{E}$ bsf ejfifsfoujbmpqfsbupst po K- uifz bsf beejujwf po K- boe E/j boe $\overline{E}/j$ bsf qpmzopn jbmt po $K^{\subseteq}$ cz efgjojujpo0 Tjodf $E ? \overline{E}$ po G- ju gpmmpxt gspn Mfn n b 607 uibu $E ? \overline{E}$ po K- boe uivt)81+ipmet gps fwfsz y / K0

Opx mfu y / K cf gjyfe0 Sfqfbujoh uif bshvn fou bcpwf xf dbo tff uibu)81+ipmet gps fwfsz x / K0 Uifsfgpsf- xf ibwf $p \times m$ / $\mathcal{T}_{(\phi,\psi)}$0 Uijt qspwft uif jn qmjdbujpo)j+? $\Leftrightarrow$)jw+0

)jw+? $\Leftrightarrow$)jjj+; Ju jt fopvhi up tipx uibu uif n bq)$x_1,\ldots,x_k$+$\not\leftarrow \bigcup_{i=1}^{k}$)$\phi_i \equiv D_i$+)$x_i$+jt pg uif gpsn $p \times m$ po)$K^{\subseteq}$+^{k}0 Cz uif frvjwbmfodf pg uif tubufn fout)jjj+boe)jw+pg Uifpsfn 608- uifsf bsf qpmzopn jbmt p_i po $K^{\subseteq}$ tvdi uibu $\phi_i \equiv D_i ? p_i \times m_i$ po K0 Uifo

$$\int_{j=1}^{k})\phi_i \equiv D_i+)x_i+? \int_{j=1}^{k} p_i)x_i+\times m_i)x_i+? \Bigg) \int_{j=1}^{k} p_i)x_i+\Big(\times m)x_1,\ldots,x_k+$$

po $K^{\subseteq}$0 Ju jt dmfbs uibu)$x_1,\ldots,x_k$+$\not\leftarrow \bigcup_{i=1}^{k} p_i)x_i$+jt b qpmzopn jbmpo)$K^{\subseteq}$+^{k}- xijdi dpn qmfuft uif qsppg0 □

Remark 6.2. Uif qsppg pg uif jn qmjdbujpo)j+? $\Leftrightarrow$)jw+hjwft uif gpmmpxjoh; jo uif sfqsf. tfoubujpo pg $p \times m$ bt b tvn pg gvodujpot pg uif gpsn)78+ uif tvn pg uif efhsfft pg uif ejfifsfoujbmpqfsbupst frvbmt uif efhsff pg p jo fwfsz ufsn 0)Uif efhsff pg b ejfifsfoujbm pqfsbups xbt efgjofe jo Tfdujpo 6020+

Theorem 6.3. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Then spectral synthesis holds in every variety on* $)K^{\subseteq}+^k$ *consisting of k-additive functions (with respect to addition).*

Proof. Tjodf K jt dpvoubcmf- tp jt uif Bcfmjbo hspvq $)K^{\subseteq}+$0 Mfu V cf b wbsjfuz po $)K^{\subseteq}+^k$ dpotjtujoh pg k.beejujwf gvodujpot0 Cz Uifpsfn 3033- mpdbm tqfdusbm tzouiftjt ipmet po $K^{\subseteq}$- boe uivt V jt tqboofe cz mpdbm qpmzopnjbm fyqpofoujbm gvodujpot0 Tjodf- cz Uifpsfn 702- fwfsz mpdbm qpmzopnjbm fyqpofoujbm gvodujpo dpoubjofe cz V jt b qpmzopnjbm fyqpofoujbm gvodujpo- ju gpmmpxt uibu V jt tqboofe cz qpmzopnjbm fyqpofoujbm gvodujpot0 □

6.2 The space of solutions of linear functional equations

Yf dpotjefs uif eftdsjqujpo pg uif tpmvujpot pg

$$\prod_{i=1}^{n} a_i f)b_i x \,, \ \ c_i y+? \ 1, \qquad)82+$$

xifsf a_i, b_i, c_i bsf hjwfo dpnqmfy ovncfst boe f ; $\mathbb{C} \Leftarrow \mathbb{C}$ jt uif volopxo gvodujpo0 Mfu K ? $\mathbb{Q})b_1, \ldots, b_n, c_1, \ldots, c_n+$0

Pvs bjn jt up hfofsbmj{f Uifpsfn 60 up uif dbtf pg $k > 2$0 Sfdbmm uibu S_k efopuft uif tfu pg uiptf tpmvujpot pg)82+ efgjofe po K xijdi bsf hfofsbmj{fe npopnjbmt pg efhsff k0 Bmtp- M_k efopuft uif tfu pg uif gvodujpot F; $K^k \Leftarrow \mathbb{C}$ tvdi uibu F jt k.beejujwf- boe uif gvodujpo $x \not\Leftarrow F)s_1x, s_2x, \ldots, s_kx+$ jt b tpmvujpo pg)82+ po K gps fwfsz $s_1, s_2, \ldots s_k$ / $K^{\subseteq}$0 Jo beejujpo-

$$M_k^{\subseteq} ? \ \}F\backslash_{(K^*)^k} \, ; F \ / \ M_k| \, .$$

Cz Mfnnbt 403 boe 404- $M_k^{\subseteq}$ jt b wbsjfuz po $)K^{\subseteq}+^k$ boe S_k ? $\}$ejbh F ; F / $M_k|$ 0

Jo Uifpsfn 4029 ju xbt qspwfe uibu jg S_k dpoubjot b opo{fsp gvodujpo- uifo uifsf bsf gjfme bvupnpsqijtnt $\phi_1, \ldots, \phi_k$ pg $\mathbb{C}$ tvdi uibu $\phi_1 \times\times\times \phi_k$ / S_k0 Uif qsppg efqfoet po uif gbdu uibu tqfdusbm bobmztjt ipmet jo b dfsubjo wbsjfuz0 Pvs qsppg pg Uifpsfn 705 jt cbtfe po uif pctfswbujpo uibu- cz Uifpsfn 704- tqfdusbm tzouiftjt ipmet jo uif tbnf wbsjfuz0

Theorem 6.4.)j+ *For every function F / M_k, F is an exponential monomial on* $)K^{\subseteq}+^k$ *if and only if F / $\mathcal{T}_{(\phi_1,\ldots,\phi_k)}$, where $\phi_1, \ldots, \phi_k$ are automorphisms of $\mathbb{C}$ and* $\bigcup_{i=1}^{k} \phi_i)x+$ *is a solution of*)82+.

)jj+ *The variety $M_k^{\subseteq}$ is spanned by the classes $M_k^{\subseteq}\{\ \mathcal{T}_{(\phi_1,\ldots,\phi_k)}$, where $\phi_1, \ldots, \phi_k$ are as above.*

Proof.)j+ Tvqqptf uibu F / M_k jt bo fyqpofoujbm npopnjbm po $)K^{\subseteq}+^k$- boe mfu F ? $p \times m$- xifsf p jt b qpmzopnjbm boe m jt bo fyqpofoujbm po $)K^{\subseteq}+^k$0 Tjodf $M_k^{\subseteq}$ jt b wbsjfuz- $p \times m$ / $M_k^{\subseteq}$

jn qmjft $m\ /\ M_k^{\subseteq}$ cz Mfnnb 30260 Opuf uibu $M_k \to V_k$ boe $M_k^{\subseteq} \to V_k^{\subseteq}$0 Uifsfgpsf- cz Mfnnb 407- uifsf bsf jokfdujwf ipnpnpsqijtnt $m_1,\ldots,m_k$ gspn K joup $\mathbb{C}$ tvdi uibu

$$m)x_1,\ldots,x_k+?\ m_1)x_1+\times\times\times m_k)x_k+\qquad)x_1,\ldots,x_k\ /\ K^{\subseteq}+.$$

Mfu ϕ_i cf bo fyufotjpo pg m_i up $\mathbb{C}$ bt bo bvupnpsqijtn pg $\mathbb{C}$0 Uifo $m)x,\ldots,x+?\ \bigcup_{i=1}^k \phi_i)x+$ jt b tpmvujpo pg)82+po K0 Uif sftu pg uif tubufnfou)j+gpmmpxt gspn Uifpsfn 7020

Tubufnfou)jj+jt b dpotfrvfodf pg Uifpsfn 7040 □

Uif tubufnfou pg uif ofyu uifpsfn gpmmpxt jnnfejbufmz gspn Mfnnb 404 boe)jj+pg Uifpsfn 7050

Theorem 6.5. *The set S_k is spanned by the classes $S_k\ \{\ \mathcal{D}_{(\phi_1,\ldots,\phi_k)}$, where $\phi_1,\ldots,\phi_k$ are automorphisms of $\mathbb{C}$, and $\bigcup_{i=1}^k \phi_i$ is a solution of*)82+. □

Mfu $S_{\subset k}$ efopuf uif tfu pg uiptf tpmvujpot pg)82+efgjofe po K xijdi bsf hfofsbmj-fe qpmzopnjbmt pg efhsff bu nptu k0 Uifo $S_{\subset k}$ jt b dmptfe mjofbs tqbdf pwfs $\mathbb{C}$0

Theorem 6.6. *The set $S_{\subset k}$ is spanned by the classes $S_m\ \{\ \mathcal{D}_{(\phi_1,\ldots,\phi_m)}$, where $1 \geq m \geq k$, $\phi_1,\ldots,\phi_m$ are automorphisms of $\mathbb{C}$, and $\bigcup_{i=1}^m \phi_i$ is a solution of*)82+.

Proof. Mfu $f\ /\ S_{\subset k}$ cf bscjusbsz0 Uifo $f\ ?\ \prod_{m=0}^k f_m$- xifsf f_m jt b hfofsbmj-fe npopnjbm pg efhsff m gps fwfsz $m\ ?\ 2,\ldots,k$- boe f_0 jt dpotubou0 Cz Mfnnb 3041- fbdi pg uif gvodujpot $f_0,\ldots,f_k$ jt b tpmvujpo pg)82+0 Uifsfgpsf- xf ibwf $f_m\ /\ S_m$ gps fwfsz $m\ ?\ 2,\ldots,k$0

Ju jt fopvhi up tipx uibu fbdi f_m jt jo uif dmptvsf pg uif mjofbs tqbdf tqboofe cz uif dmbttft $\mathcal{D}_{(\phi_1,\ldots,\phi_m)}\ \{\ S_m$- xifsf $\phi_1,\ldots,\phi_m$ bsf bvupnpsqijtnt pg $\mathbb{C}$- boe $\bigcup_{i=1}^m \phi_i)x+/\ S_m$0 Jg $m \subset 2$ uifo uijt jt usvf cz Uifpsfn 7060

Jg $m\ ?\ 1$- uifo uifsf bsf uxp dbtft up dpotjefs0 Jg $\prod_{i=1}^n a_i\ ?\ 1$- uifo uif pomz dpotubou tpmvujpo pg)82+jt uif -fsp gvodujpo- tp $f_0\ ?\ 10$ Po uif puifs iboe- jg $\prod_{i=1}^n a_i\ ?\ 1$- uifo bmm dpotubou gvodujpot bsf tpmvujpot pg)82+0 Uifo uif tubufnfou jt usvf- tjodf $\mathcal{D}_{\mathcal{D}}$ jt uif dmbtt pg dpotubou gvodujpot0 □

Xf tibmm tbz uibu uif frvbujpo)82+jt *normal of degree* k- jg fwfsz tpmvujpo pg)82+jt b hfofsbmj-fe qpmzopnjbm pg efhsff bu nptu k0 Opx xf hjwf b eftdsjqujpo pg uif tfu pg tpmvujpot pg opsnbm frvbujpot0

Corollary 6.7. *Suppose that the equation*)82+*is normal of degree* k. *Then the linear space of its solutions defined on K is spanned by the classes $S_m\ \{\ \mathcal{D}_{(\phi_1,\ldots,\phi_m)}$, where $1 \geq m \geq k$, $\phi_1,\ldots,\phi_m$ are automorphisms of $\mathbb{C}$, and $\bigcup_{i=1}^m \phi_i$ is a solution of*)82+.

In particular, this is true for the equation

$$\prod_{i=1}^{n} a_i f)b_i x \,,\; y+?\; 1 \qquad)x,y \;/\; \mathbb{C}+ \qquad)83+$$

with k ? n 3. □

Theorem 6.8. *Let* S *denote the set of solutions of*)82+*defined on* K.

)j+ *If the equation is normal, then the set of polynomials on the additive group of* K *is dense in* S.

)jj+ *If the equation is normal and* $\prod_{i=1}^{n} a_i$? 1, *then spectral synthesis holds in* S *considered as a variety on the additive group of* K.

)jjj+ *If the equation is normal and translation invariant, then spectral synthesis holds in* S *considered as a variety on the additive group of* K.

Proof. Jg ϕ jt bo bvupn psqi jtn pg $\mathbb{C}$- boe D jt b ejffsfoujbmpqfsbups po K- uifo uif gvodujpo $\phi \equiv D$ jt beejujwf po K0 Uifsfgpsf- fbdi dmbtt $\mathcal{D}_{(\phi_1,\dots,\phi_k)}$ dpotjtut pg qpmzopnjbmt0 Uivt)j+gpmmpxt gspn Dpspmmbsz 7080

Mfu 1 efopuf uif jefoujdbmmz 2 gvodujpo efgjofe po K0 Jg $\prod_{i=1}^{n} a_i$? 1- uifo 1 jt b tpmvujpo0 Tjodf 1 jt bo fyqpofoujbmgvodujpo- fwfsz qpmzopnjbmjo S jt- jo gbdu- b qpmzopnjbm fyqpofoujbmgvodujpo0 Uivt)jj+gpmmpxt gspn)j+0

Tvqqptf uibu uif frvbujpo jt opsnbmboe usbotmbujpo jowbsjbou0 Jg S ? }1|- uifo tqfdusbmtzouiftjt ipmet jo S0 Jg S dpoubjot b opo–fsp hfofsbmj-fe qpmzopnjbm$p$- uifo ju gpmmpxt gspn uif usbotmbujpo jowbsjbodf pg S uibu $\Lambda_{h_1}\dots\Lambda_{h_k}p$ / S gps fwfsz k boe $h_1,\dots,h_k$ / $\mathbb{C}$0 Jg k ? efh p- uifo $\Lambda_{h_1}\dots\Lambda_{h_k}p$ jt b opo–fsp dpotubou gps tvjubcmf $h_1,\dots,h_k$0 Uivt 1 / S- boe uivt)jjj+gpmmpxt gspn)jj+0 □

6.3 An example and concluding remarks

Uif gpmmpxjoh fybnqmf tfswft bt bo jmmvtusbujpo; ju tipxt ipx up vtf pvs qsfwjpvt sftvmut jo psefs up efufsnjof uif tfu pg tpmvujpot pg b hjwfo frvbujpo0 Xf dpotjefs uif frvbujpo

$$\begin{aligned} &3f)y+, \; f)tx \,,\; y+, \; f))2 \quad t+x \,,\; y+, \\ &, \; f))t^2 \quad t+x \,,\; y+ \quad f))t^2 \quad t \,,\; 2+x \,,\; y+?\; 1, \end{aligned} \qquad)84+$$

xifsf t jt b gjyfe usbotdfoefoubmovncfs0 Uif frvbujpo jt pg uif gpsn)83+xifsf n ? 6- $)a_1,\dots,a_5+?$) $3,2,2,2,$ $2+$boe $)b_1,\dots,b_5+?$ $)1,t,2$ t,t^2 t,t^2 t , $2+$0 Uivt fwfsz tpmvujpo jt b hfofsbmj-fe qpmzopnjbmpg efhsff bu nptu uisff0

Jo psefs up tjn qmjgz opubujpo- xf ti bmm xsjuf x^{∞} jotufbe pg $\frac{\partial}{\partial t}x$- $x^{\infty\infty}$ jotufbe pg $\frac{\partial^2}{\partial t^2}x$ fud0 Ju jt fbtz up difdl ui bu xf i bwf

$$\prod_{i=1}^{5} a_i \times b_i^n ?\ 1\ )n ?\ 1,2,3+, \qquad \prod_{i=1}^{5} a_i \times b_i^3 \not?\ 1. \qquad)85+$$

Y f ti bmm bmtp offe

$$\prod_{i=1}^{5} a_i \times b_i^{(n)} \times b_i^{(m)} ? \begin{cases} 3 & \text{jg}\, n ?\ m ?\ 2, \\ 3 & \text{jg}\, n ?\ 3,\ m ?\ 1, \\ 1 & \text{jg}\, n \subset m \subset 1,\ n,\ m \not?\ 3. \end{cases} \qquad)86+$$

Ju jt fbtz up tff ui bu ui f frvbujpo jt usjwjbm-ui bu jt- fwfsz beejujwf gvodujpo jt b tpmvujpo0)Tjodf $a_1, \ldots, a_5$ bsf joufhfst boe K ? $\mathbb{Q})t+$ jt qvsfmz usbotdfoefoubm ju gpmmpx t gspn Ui fpsfn 6025 ui bu jg)84+ x fsf opu usjwjbm ju x pvme o(u i bwf boz opo–fsp tpmvujpo po K0+ Tjodf ui f n bqt $x \not\Leftarrow x^{(n)}$ bsf beejujwf gvodujpot- xf i bwf

$$\prod_{i=1}^{5} a_i \times b_i^{(n)} ?\ 1 \qquad)n ?\ 2,3,\ldots+. \qquad)87+$$

Mfu ϕ, ψ cf bvupn psqi jtn t pg $\mathbb{C}$- boe tvqqptf ui bu g ? $\phi \times \psi$ jt b tpmvujpo pg)84+0 Jg xf tvctujuvuf f ? g, y ? 1 boe x ? 2 joup)84+ ui f mfgu i boe tjef pg ui f frvbujpz pcubjofe frvbmt $\phi)t+^2$, $\psi)t+^2$ $3\phi)t+\psi)t+?$ $)\phi)t+$ $\psi)t+)^2$0 Tjodf g jt b tpmvujpo- xf hfu $\phi)t+?$ $\psi)t+$. Ui jt jn qmjft ui bu ϕ ? ψ po K0

Y f qspwf ui bu ui fsf jt op opo–fsp tpmvujpo x i jdi jt b hfofsbmj–fe n popn jbm pg efhsff 40 Joeffe- tvqqptf ui fsf jt tvdi b tpmvujpo0 Ui fo ju gpmmpx t gspn Ui fpsfn 706 ui bu ui fsf bsf bvupn psqi jtn t ϕ, ψ, χ pg $\mathbb{C}$ tvdi ui bu $\phi \times \psi \times \chi$ jt b tpmvujpo0 Ui fo- cz Ui fpsfn 4032- ui f gvodujpot $\phi \times \psi$, $\phi \times \chi$, $\psi \times \chi$ bsf bmtp tpmvujpot0 Bt xf tbx bcpwf- ui jt jn qmjft ui bu ϕ ? ψ ? χ po K- boe ui vt ϕ^3 jt b tpmvujpo0 Ui fo xf i bwf

$$1 ?\ \prod_{i=1}^{5} a_i \times \phi)b_i+^3 ?\ \phi \Big) \prod_{i=1}^{5} a_i \times b_i^3 \Big(,$$

x i jdi dpousbejdut)85+0 Ui jt qspwft ui bu ui fsf jt op opo–fsp tpmvujpo x i jdi jt b hfofsbmj–fe n popn jbm pg efhsff 40

Ui fsfgpsf- jo psefs up efufsn jof bmm tpmvujpot pg)84+ ju jt fopvhi up eftdsjcf ui f tfu S_2 pg ui ptf tpmvujpot x i jdi bsf efÞofe po K boe bsf hfofsbmj–fe n popn jbmt pg efhsff ux p0

Jg ϕ, ψ bsf bvupn psqi jtn t pg $\mathbb{C}$- ui fo $\mathcal{D}_{(\phi,\psi)}$ efopuft ui f tfu pg gvodujpot pg ui f gpsn $\prod_{j=1}^{N})\phi \equiv D_j+\times)\psi \equiv E_j+$ x i fsf D_j boe E_j bsf ejggfsfoujbm pqfsbupst po K0 Cz Ui fpsfn 706- ui f tfu S_2 jt tqboofe cz S_2 { $\mathcal{D}_{(\phi,\psi)}$- x i fsf ϕ, ψ bsf bvupn psqi jtn t pg $\mathbb{C}$ tvdi ui bu

$\phi \times \psi$ jt b tpmvujpo pg)84+0 Tjodf ui jt jn qmjft ϕ ? ψ- xf n bz dpoffiof pvs buufoujpo up ui f tfut $\mathcal{D}_{(\phi,\phi)}$0 Ju jt dmfibs ui bu

$$\mathcal{D}_{(\phi,\phi)} \ ? \ \phi \ \ \mathcal{D}_{(j,j)}\Big($$

gps fwfsz bvupn psqi jtn ϕ- xi fsf j efopuft ui f jefoujuz n bq0 Bmtp- f jt b tpmvujpo pg)5: + jg boe pomz jg $\phi \equiv f$ jt- tp xf pomz offe up eftdsjcf S_2 { $\mathcal{D}_{(j,j)}$0

Tjodf fwfsz ejfifsfoujbmpqfsbups po K ? $\mathbb{Q})t+$ jt ui f mjofbs dpn cjobujpo pg ui f n bqt $x \not\Leftarrow x^{(n)}$)x / K, n ? $1, 2, \ldots$+ ui f fmfn fout pg $\mathcal{D}_{(j,j)}$ bsf mjofbs dpn cjobujpot pg ui f n bqt

$$f_{n,m})x+? \ x^{(n)} \times x^{(m)} \qquad)x \ / \ K, \ 1 \geq m \geq n+$$

Jo psefs up efufsn jof xi jdi mjofbs dpn cjobujpot pg ui f n bqt $f_{n,m}$ bsf tpmvujpot- xf i bwf up dpn qvuf ui f tvn t

$$S_{n,m})x, y+? \ \prod_{i=1}^{5} a_i \times f_{n,m})b_i x \ , \ \ y+,$$

boe efufsn jof ui ptf mjofbs dpn cjobujpot pg ui f gvodujpot $S_{n,m}$ xi jdi bsf jefoujdbmmz -fsp po K ? $\mathbb{Q})t+$0 B dpn qvubujpo- cbtfe po)86+boe)87+ ti px t ui bu xf i bwf $S_{0,0}$? $S_{1,0}$? $S_{1,1}$, $S_{2,0}$? 1 po K- boe jg b mjofbs dpn cjobujpo pg ui f gvodujpot $S_{n,m}$ jt -fsp po K- ui fo ju jt bmtp b mjofbs dpn cjobujpo pg $S_{0,0}$, $S_{1,0}$ boe $S_{1,1}$, $S_{2,0}$0

Joeffe- xf i bwf

$$)b_i x \ , \ \ y+^{(n)} \ ? \ y^{(n)} \ , \ \prod_{\nu=0}^{n} \Big)\begin{matrix} n \\ \nu \end{matrix}\Big(b_i^{(\nu)} x^{(n \ \ \nu)} \ ? \ y^{(n)} \ , \ \prod_{\nu=0}^{2} \Big)\begin{matrix} n \\ \nu \end{matrix}\Big(b_i^{(\nu)} x^{(n \ \ \nu)},$$

boe

$$)b_i x \ , \ \ y+^{(m)} \ ? \ y^{(m)} \ , \ \prod_{\mu=2}^{n} \Big)\begin{matrix} m \\ \mu \end{matrix}\Big(b_i^{(\nu)} x^{(m \ \ \mu)}.$$

Ubl joh ui f qspevdut pg ui f sjhi u i boe tjeft- n vmujqmzjoh cz a_i- tvn n joh gps i ? $2, \ldots, 6$ boe vtjoh)85+-)86+ xf pcubjo

$$\begin{aligned} S_{n,m})x+? \ & \prod_{\nu,\mu=0}^{2} \Big)\begin{matrix} n \\ \nu \end{matrix}\Big)\begin{matrix} m \\ \mu \end{matrix}\Big(x^{(n \ \ \nu)} x^{(m \ \ \mu)} \times \prod_{i=1}^{5} a_i \times b_i^{(\nu)} b_i^{(\mu)} \ ? \\ ? \quad & 3\Big)\begin{matrix} m \\ 3 \end{matrix}\Big(x^{(n)} x^{(m \ \ 2)} \ , \ \ 3nm \times x^{(n \ \ 1)} x^{(m \ \ 1)} \quad 3\Big)\begin{matrix} n \\ 3 \end{matrix}\Big(x^{(n \ \ 2)} x^{(m)}. \end{aligned} \qquad)88+$$

Tvqqptf f ? $\prod_{(n,m)\emptyset I} c_{n,m} \times f_{n,m}$ jt b tpmvujpo- xi fsf ui f dpfı djfout $c_{n,m}$ bsf opo-fsp gps fwfsz)n, m+/ I0 Ui fo

$$\prod_{(n,m)\emptyset I} c_{n,m} \times S_{n,m})x+? \ 1$$

gps fwfsz x / $\mathbb{Q})t+$0 Opx xf vtf ui f gbdu ui bu gps fwfsz ffiojuf tfu I pg qbjst)n, m+ui fsf fyjtut b qpmzopn jbmp / $\mathbb{Q}]t^{\text{`}}$ tvdi ui bu ui f qpmzopn jbmt $p^{(n)} \times p^{(m)}$))n, m+/ I+bsf mjofbsmz joefqfoefou pwfs ℂ0)Yf pn ju ui f qspp0+ Ui jt jn qmjft ui bu jg jo ui f tvn $\prod_{(n,m)\emptyset I} c_{n,m} \times S_{n,m})x+$xf sfqmbdf fbdi $S_{n,m})x+$cz ui f sjhi u i boe tjef pg ui f frvbujpo)88+ boe sfqmbdf x cz $p)t+$ ui fo xf pcubjo –fsp pom jg ui f frvbujpo pcubjofe jt bo jefoujuz=ui bu jt- jgfwfsz ufsn $x^{(i)} \times x^{(j)}$ jt dbodfmfe0

Tvqqptf I dpoubjot b qbjs $)n, m+$x jui $m \subset 30$ Mfu $)n, m+$cf tvdi b qbjs x jui ui f mbshftu n boe- jg ui fsf bsf npsf ui bo pof tvdi qbjs ui fo x jui b mbshftu $m \subset 30$ Ju jt dmfbs ui bu jo ui jt dbtf ui f ufsn $x^{(n)} \times x^{(m\ \ 2)}$ pddvst pomz podf jo ui f tvn x jui b opo–fsp dpf1 djfou- tp ju jt opu dbodfmfe pvu0

Ui fsfgpsf- xf i bwf m ? 1 ps m ? 2 gps fwfsz $)n, m+/ I0$ Yf i bwf

$$S_{(n,0)} ?\ n)n\quad 2+\times x^{(n\ \ 2)} \times x, \qquad S_{(n,1)} ?\ 3n \times x^{(n\ \ 1)} \times x \quad n)n \quad 2+\times x^{(n\ \ 2)} \times x^{\infty}$$

gps fwfsz $n \subset 30$ Jg I dpoubjot b qbjs $)n, 2+$x jui $n \subset 3$- ui fo ui f tvn $\prod_{(n,m)\emptyset I} c_{n,m} \times S_{n,m})x+$ x jmmdpoubjo b ufsn $x^{(n\ \ 2)} \times x^{\infty}$ui bu pddvst pomz podf x jui b opo–fsp dpf1 djfou- x i jdi jt jn qpttjcmf0 Ui fsfgpsf- ui f pomz qbjs $)n, m+/ I$ x jui m ? 2 jt $)2, 2+0$ Ui fo- jg I dpoubjot b qbjs $)n, 1+$x jui $n \subset 4$- ui fo ui f tvn $\prod_{(n,m)\emptyset I} c_{n,m} \times S_{n,m})x+$x jmmdpoubjo b ufsn $x^{(n\ \ 2)} \times x$ ui bu pddvst pomz podf x jui b opo–fsp dpf1 djfou- x i jdi jt jn qpttjcmf0 Ui fsfgpsf- ui f pomz qbjs $)n, m+/ I$ x jui m ? 2 jt $)2, 2+0$

Ui jt n fbot ui bu S_2 { $\mathcal{D}_{(j,j)}$ jt ui f mjofbs tqbo pg ui f gvodujpot $x^2, x \times x^{\infty}$boe $)x^{\infty}+^2,\ x \times x^{\infty\infty}0$ Tvn n joh vq; *The space of solutions of*)84+*defined on* K *is the closed linear hull of all additive functions and the functions*

$$\phi^2,\ \phi \times \Big) \phi \equiv \frac{\partial}{\partial t}\Big(\text{ and } \Big)\phi \equiv \frac{\partial}{\partial t}\Big(^2,\ \phi \times \Big)\phi \equiv \frac{\partial^2}{\partial t^2}\Big(,$$

where ϕ is an arbitrary injective homomorphism of K.

Ui f fybn qmf bcpwf ti px t ui bu tpn f pg ui f sftvmut pg Tfdujpo 505 dboopu cf hfofsbmj–fe gps tpmvujpot pg efhsff hsfbufs ui bo 20 Ui fpsfn 6021 tbzt ui bu jg $a_1, \ldots, a_n$ bsf bmhfcsbjd ovn cfst boe ui f jokfdujwf i pn pn psqi jtn ϕ jt b tpmvujpo- ui fo $\phi \equiv D$ jt bmtp b tpmvujpo gps fwfsz ejggfsfoujbmpqfsbups $D0$ Jo ui f fybn qmf bcpwf- $\phi \times \phi$ jt b tpmvujpo gps fwfsz ϕ- cvu $)\phi \equiv \frac{\partial}{\partial t}+^2$ jt opu b tpmvujpo- tp ui f bobmphz jt gbmtf gps n popn jbmt pg efhsff 30 Ui jt jn qmjft ui bu ui f bobmphz pg Dpspmmbsz 708 jt bmtp gbmtf gps n popn jbmt pg efhsff 30

Ui fpsfn 6024 tbzt ui bu jg b_i, c_i / $\mathbb{Q})t+$ x i fsf t jt usbotdfoefoubmpwfs ℚ- ui fo ui f frvbujpo jt fjui fs usjwjbmps S_1 jt pg ggojuf ejn fotjpo pwfs ℂ0 Ui f bobmphpvt tubufn fou xpvme cf ui bu jg b_i, c_i / $\mathbb{Q})t+$ ui fo fjui fs fwfsz n popn jbmpg efhsff uxp jt b tpmvujpo- ps S_2 jt pg ggojuf ejn fotjpobm0 Ui f fybn qmf bcpwf ti px t ui bu ui jt jt opu usvf jo hfofsbm0 Yf dbo tff ui bu ui f bobmphvf pg Ui fpsfn 6025 jt bmtp gbmtf jo $S_2$0

Yf sfn bsl- i pxfwfs- ui bu *if the space of additive solutions of an equation*)83+*is of finite dimensional, then so is the space of those solutions which are generalized monomials* *of degree two.*)Ui jt gpmmpxt gspn ui f gbdu ui bu jg $A)x, y+$jt tzn n fusjd boe cjbeejujwf-boe x ∀= $A)x, x+$jt b tpmvujpo- ui fo ui f gvodujpot y ∀= $A)x, y+)x$ / $K+$bsf beejujwf tpmvujpot0 Jg ui ftf mbuufs gvodujpot tqbo b mjofbs tqbdf pg ffiojuf ejn fotjpo pwfs ℂ hfofsbufe cz ui f beejujwf gvodujpot $a_1, \ldots, a_k$- ui fo $A)x, y+$jt ui f mjofbs dpn cjobujpo pg ui f gvodujpot $a_i)x+\times a_j)y+)i, j$? $2, \ldots, k$++

Bmui pvhi ui f eftdsjqujpo pg ui f tfu pg tpmvujpot pg b hjwfo frvbujpo dbo cf ejı dvm- ui f fybn qmf bcpwf ti pxt ui bu- bu mfbtu jo qsjodjqmf- ui f eftdsjqujpo jt qpttjcmf jo ui f dbtf pg n boz frvbujpot0

Y f dpodmvef xjui tpn f sfn bslt dpodfsojoh ui f ahfofsjd(ps asboepn (frvbujpo0 Cz ui bu xf n fbo bo frvbujpo)82+jo xi jdi ui f ovn cfst a_i, b_i, c_i bsf bmhfcsbjdbmmz joefqfoefou pwfs Q0 Tvdi bo frvbujpo jt opsn bmmz cvu opu usbotmbujpo jowbsjbou0 Bo jokfdujwf i pn pn psqi jtn ϕ jt b tpmvujpo jg boe pomz jg

$$\prod_{i=1}^{n} a_i\phi)b_i+?\ \prod_{i=1}^{n} a_i\phi)c_i+?\ 1 \qquad)89+$$

i pmet0 Ui jt jn qmjft ui bu ui f frvbujpot jt opu usjwjbm)opu fwfsz beejujwf gvodujpo jt b tpmvujpo+ cvu S_1 jt pg joffiojuf ejn fotjpobmju0 Pof dbo qspwf ui bu S_1 *is spanned by the injective homomorphisms satisfying*)89+. Opuf ui bu ejfifsfoujbm pqfsbupst ep opu bqqfbs jo ui f eftdsjqujpo pg $S_1$0

7 The discrete Pompeiu problem

Jo ui jt tfdujpo xf bsf dpodfsofe xjui ui f tp.dbmmfe ejtdsfuf Qpn qfjv qspcmfn boe jut dpo. ofdujpo up mjofbs gvodujpobmfr vbujpot0 Ui f qspcmfn - up cf fyqmbjofe ti psumz- jt tufn n fe gspn ui f dmbttjdbmQpn qfjv qspcmfn boe gspn ui f gpmmpxjoh r vftujpo btl fe cz MD Qǒtb0

Question 7.1)Qǒtb+. *Suppose that the function* $f; \mathbb{R}^2 \Leftarrow \mathbb{R}$ *has the property that the sum of the values of* f *at the vertices of any square of fix size is zero. Is it true that* $f \subseteq 1$*?*

Bt xf ti bmmtff- ui f botxfs up Qǒtb(t r vftujpo jt bı sn bujwf0 Ui jt r vftujpo dbo cf hfofsbmj-fe bt gpmmpxt0

Question 7.2)Ejtdsfuf Qpn qfjv qspcmfn +. *Let* $D \to \mathbb{R}^2$ *be a finite set. Suppose that the function* $f; \mathbb{R}^2 \Leftarrow \mathbb{R}$ *has the property that the sum of the values of* f *at the the points of every congruent copy of* D *is zero. Is it true that* $f \subseteq 1$*?*

Ui f dmbttjdbmQpn qfjv qspcmfn jt ui f gpmmpxjoh r vftujpo jo jouf hsbmhf pn fusz- obn fe bgufs Ejn jusjf Qpn qfjv0

Question 7.3. *Let* f *be a continuous function defined on the plane, and let* K *be a closed set of positive Lebesgue measure. Suppose that*

$$\int_{\sigma(K)} f)x, y+dxdy \; ? \; 1 \qquad)8:+$$

for every rigid motion σ*. Is it true that* $f \subseteq 1$*?*

Yf tbz ui bu ui f tfu K i bt ui f *Pompeiu's property* jg ui f botxfs up ui f Rvftujpo 804 jt bı sn bujwf0

Jo ui f dbtf xi fsf K jt b ejtd- Qpn qfjv bttfsufe ui bu ui f botxfs xbt bı sn bujwf boe fwfo qvcmjti fe bo fsspofpvt qsppg0 Ui f fssps xbt qpjoufe pvu cz Ojdpmftdp- xi p tpvhi u up ftubcmjti hfofsbmj-bujpot pg Qpn qfjv(t sftvmu0 Di bl bmpw]5' ti pxfe ui bu ui fsf bsf opousjwjbm tpmvujpot up)8: +-gps jotubodf- ui fsf bsf jogjojufmz n boz mjofbsmz joefqfoefou tpmvujpot pg ui f gpsn tjo)ax , by+gps bqqspqsjbufmz di ptfo dpotubout a, b0

Jo dbtf K jt b tr vbsf- Qpn qfjv ti pxfe ui bu ui f pomz dpoujovpvt tpmvujpo up)8: + ufoejoh up b mjn ju bu jogjojuz jt ui f -fsp gvodujpo0 Jo gbdu- i jt qsppg jt n psf dpn qmjdbufe ui bo ofdfttbsz- boe i f bduvbmmz offet pomz ui f gbdu ui bu)8: +i pmet gps bmmtr vbsft pg b gjyfe tj-fe qbsbmmfmup ui f dppsejobuf byft0 Di sjtupw]6'-]7' ti pxfe ui bu Qpn qfjv(t sfr vjsfn fou ui bu f ufoe up b mjn ju dpvme cf espqqfe boe tvctfr vfoumz tfuumfe ui f dpssftqpoejoh qspcmfn gps qbsbmmfmphsbn t boe gps usjbohmft0

Jo hfofsbmM0 Cspxo- C0 N0 Tdi sfjcfs boe C0 B0 Ubzmps]4' ti pxfe ui bu ui f epn bjo K i bt ui f Qpn qfjv(t qspqfsuz xi fo K jt boz qpmzhpobmsfhjpo ps boz dpowfy tfu xjui bu rfibtu pof ’dpsofs’0)Gps n psf efubjmt tff]4- Dpspmbsz 6023‘0+ Ui fjs qsppgjt cbtfe po ui f gbdu ui bu tqf dusbmtzoui ftjt i pmt jo ui ptf wbsjfujft pg dpoujovpvt gvodujpot po $\mathbb{R}^n$ x i jdi bsf jowbsjbou voefs usbotmbujpot boe spubujpot0

Ui f qspcmfn i bt n boz wbsjbout0 Ui f uzqjdbmpoft vtf tjn jmbs dpqjft ps usbotmbuft pg K jotufbe pg dpohsvfoudpqjft0 Pof uzqf pgui f ejtdsfuf wfstjpo pg Qpn qfjv qspcmfn bqqfbsfe jo ui f qbqfs pg E0[fjmfshfs]58‘0 Y f efopuf cz $\mathbb{Z}^n$ ui f n.ejn fotjpobmmbuujdf jo $\mathbb{R}^n$0 Mfu $\mathcal{E}$ cf b ffojuf gbn jmz pg ffojuf tvctfut pg $\mathbb{Z}^n$- boe mfu $\cap$ efopuf ui f hspvq pg bmmusbotmbujpot po $\mathbb{Z}^n$0 Y f efopuf wbsjbcmft $)z_1, z_1^{\;1}, \ldots, z_n, z_n^{\;1}+$cz z- boe ui f tfu pg dpn qmfy wbmvfe gvodujpo po $\mathbb{Z}^n$ cz $\mathcal{M})\mathbb{Z}^n+$0 Y f tbz ui bu ui f gbn jmz $\mathcal{E}$? $\}D_1, \ldots, D_k|$ i bt ui f *discrete Pompeiu's property on* $\mathbb{Z}^n$ jg ui f pomz gvodujpo $f ; \mathbb{Z}^n \Leftarrow \mathbb{C}$ tvdi ui bu

$$\prod_{d\emptyset \tau(D)} f)d+? \ 1 \text{ gps bmm } \tau \ / \cap \text{ boe } D \ / \mathcal{E} \qquad)91+$$

jt jefoujdbmmz –fsp0 [fjmfshfs qspwfe ui f gpmmpxjoh ui fpsfn ;

Theorem 7.4. *For* K *a finite subset of* $\mathbb{Z}^n$ *let* $P_D)z+? \prod_{d\emptyset\{} \mathbf{z}^d$*. Then the finite family* $\mathcal{E}$ *has the discrete Pompeiu's property on* $\mathbb{Z}^n$ *if and only if the polynomials* $\}P_D ; D \ / \mathcal{E}|$ *have common zeros in* $\mathbb{C}^n$*.*

Ui f qsppgjt cbtfe po ui f dpoofdujpo cfuxffo ejffsfodf pqfsbupst boe ui f qpmzopn jbmt po $\mathbb{C}$- boe vtft I jmcfsu(t *Nullstellensatz*]56- q0 268‘ boe pui fs sjoh ui fpsfujdbmsftvmut0 Sfdfoumz- N 0K0Qvmb]37‘ hbwf b ofdfttbsz boe tvı djfou dpoejujpo gps b ffojuf dpmmfdujpo pg ffojuf tvctfut pg b ejtdsfuf Bcfmjbo hspvqt- x i ptf upstjpo gsff sbol jt mftt ui bo dpoujovvn - up i bwf ui f Qpn qfjv(t qspqfsuz0

Y f n bz hfofsbmj–f ui f ejtdsfuf Qpn qfjv qspcmfn bt gpmmpxt0 Mfu D cf b ffojuf tfu pg $\mathbb{R}^2$ boe mfu G cf b usbotgpsn bujpo hspvq po $\mathbb{R}^2$0 Y f tbz ui bu D i bt ui f *discrete Pompeiu property with respect to* G jg gps fwfsz gvodujpo $f ; \mathbb{R}^2 \Leftarrow \mathbb{C}$ ui f frvbujpo

$$\prod_{d\emptyset \sigma(D)} f)d+? \ 1 \qquad)92+$$

gps bmm $\sigma \ / G$ jn qmjft $f \subseteq 1$0

Pvs n bjo dpodfso jt ui f botxfs up Qǒtb(t rvftujpo x i fo D jt ui f wfsujdft pg b trvbsf boe G jt ui f hspvq pg dpohsvfodft pg $\mathbb{C}$0

I pxfwfs- bt b n pujwbujpo- x f ti px ui bu boz ffojuf D i bt ui f ejtdsfuf Qpn qfjv(t qspq. fsuz xjui sftqfdu up ui f tjn jmbsjujft pg $\mathbb{C}$0 Y f efopuf cz Φ ui f tjn jmbsjuz hspvq pg $\mathbb{C}$0

Proposition 7.5. *Suppose that* D *is a nonempty finite subset of* $\mathbb{C}$*. Let* $f ; \mathbb{R}^2 \Leftarrow \mathbb{C}$ *be a function which satisfies equation*)92+ *for every* $\sigma \ / \Phi$*. Then* $f \subseteq 1$*.*

Proof. Mfu $d_1, d_2, \ldots, d_n$ / ℂ cf uif fmfnfout pg D0 Uifo uif frvbujpo)92+ dbo cf xsjuufo jo uif gpsn

$$f)x\,,\ d_1y+,\ \ f)x\,,\ d_2y+,\ \ \ldots,\ \ f)x\,,\ d_ny+?\ 1 \qquad)93+$$

ipmet gps fwfsz x, y / ℂ0 Uivt- cz Sfnbsl 4023- uif frvbujpo)93+ ibt b tpmvujpo jg boe pomz jg ju ibt bo bvupnpsqijtn tpmvujpo ϕ xijdi tbujtgjft uif frvbujpot $\prod_i a_i$? 1 boe $\prod_i a_i\phi)b_i+$? 10 Tjodf uif gjstu frvbujpo epft opu ipme- uifsfgpsf uif tfu D ibt uif ejtdsfuf Qpnqfjv(t qspqfsuz xjui sftqfdu up uif tjnjmbsjujft0 □

Gspn opx po xf uvso up Qǒtb(t rvftujpo)tff Uifpsfn 80+0

Yf xjmm vtf tpnf xfmm lopxo sftvmut pg Fvdmjefbo Sbntfz uifpsz0 Uif Fvdmjefbo Sbntfz uzqf rvftujpo jt uif gpmmpxjoh; Mfu S cf b tfu pg qpjout jo $\mathbb{R}^n$ boe mfu G cf b usbotgpsnbujpo hspvq pg $\mathbb{R}^n$0 Dpmps uif qpjout pg uif tqbdf $\mathbb{R}^n$ xjui k dpmpst0 Jt ju usvf uibu uifsf jt b npopdispnbujd dpqz $g)S+$ pg S- xifsf g / GA Jg ju jt usvf gps boz dpmpsjoh pg $\mathbb{R}^n$ xjui k dpmpst- uifo xf tbz uibu S jt Sbntfz xjui sftqfdu up G boe k dpmpst0 Jg G jt uif hspvq pg dpohsvfodft pg $\mathbb{R}^n$- uifo xf kvtu tbz S jt Sbntfz0 M0F0Ti befs jo ijt qbqfs]41‘ tipxfe uif gpmmpxjoh uifpsfn

Theorem 7.6.)j+ *For any 3-coloring of the plane all right triangles are Ramsey.*

)jj+ *For any 3-coloring of the plane and for every nondegenerate parallelogram P, there is a congruent parallelogram with three vertices of the same color.*

Yf xjmm vtf uif gpmmpxjoh xfmm lopxo tfmfdujpo uifpsfn pg Sbep)gps efubjmt tff]39‘-]48- Uifpsfn B‘+0

Theorem 7.7)Sbep(t Tfmfdujpo Qsjodjqmf+. *Let $\mathcal{D}$? $\}A_i$; i / $I|$ be a family of finite sets. For every finite $J \rightarrow I$, let f_J be a function defined on J such that $f_J)i+$ / A_i for every i / J. Then, there exists a function f defined on I such that for each finite $J \rightarrow I$ there is a finite $K \rightarrow I$ such that $J \leq K$ and $f\backslash_J$? $)f_K+\backslash_J$.*

Remark 7.8. Yf vtf uijt uifpsfn gps dpmpsjoht0 B dpmpsjoh pg b tfu A xjui $k <$ ∈ dpmpst jt b dipjdf gvodujpo0 Uivt- xf nbz vtf ju gps Sbntfz uzqf ps Fvdmjefbo Sbntfz uzqf uifpsfnt0 Gspn Uifpsfn 808 ju gpmmpxt uibu jg uifsf jt b tfu S xijdi jt Sbntfz- uifo uifsf fyjtut b gjojuf tfu pg qpjout R tvdi uibu fwfsz k.dpmpsjoh pg R dpoubjot b npopdispnbujd dpqz pg S0 Yf sfgfs up R bt b xjuoftt tfu up S0

Qǒtb(t rvftujpo xbt botxfsfe joefqfoefoumz cz U0Ufsqbj boe N0Mbd-lpwjdi boe uif bvuipst0 Uif mbuufs qsppg bduvbmmz hjwft npsf=tff Sfnbsl 80210

Theorem 7.9. *Let D be the vertex set of the unit square. Then D has the discrete Pompeiu property with respect to the congruences of ℂ.*

Proof. Ui f tubufn fou dbo cf xsjuufo jo ui f gpmmpx joh gpsn ;

$$f)x+, \quad f)x\ ,\ \ y+, \quad f)x\ ,\ \ iy+, \quad f)x\ ,\ \ )2\ ,\ \ i+y+?\ 1 \qquad)94+$$

i pmet gps fwfsz x, y / $\mathbb{C}$ boe \y\? 20 Mfu vt bttvn f ui bu ui fsf fyjtut b opo–fsp f tbujtgzjoh)94+0

Opx xf x pvme mjlf up vtf Ui fpsfn 3029 cvu vogpsuvobufmz $r_0)H+$ ui f upstjpo gsff sbol pg ui f beejujwf hspvq H pg $\mathbb{C}$ jt dpoujovvn 0 Ui fsf gpsf xf sftusjdu pvs buufoujpo up b tvc hspvq G hfofsbufe cz dpvoubcmfi n boz hfofsbupst pg H0

Jo Ui fpsfn 807 ju x bt qspwfe ui bu bmmsjhi u usjbohmft bsf S bn tfz0 Mfu S cf ui f frvj. mbufsbmsjhi u usjbohmfi pg voju tjef0 Cz S fn bsl 809- xf dbo ffiy b ffiojuf x juoftt tfu R pg S0

Tjodf $f \subseteq 1$- ui fsf jt bo a / $\mathbb{C}$ tvdi ui bu $f)a+$? 10 Mfu G efopuf ui f beejujwf tvc hspvq pg $\mathbb{C}$ hfofsbufe cz ui f fmfn fout pg R boe a0 Ui fo G jt bo Bcfmjbo hspvq pg ffiojuf upstjpo gsff sbol 0 Mfu V efopuf ui f tfu pg gvodujpot $f; G \Leftarrow \mathbb{C}$ tbujtgzjoh)94+ gps fwfsz x, y tvdi ui bu $x, x\ ,\ \ y, x\ ,\ \ iy, x\ ,\ \ )2\ ,\ \ i+y$ / G boe \y\? 20 Ju jt fbtz up tff ui bu V jt b wbsjfuz0

Tjodf $f\backslash_G$ / V boe a / G- ju gpmmpx t ui bu V_G i? 10 Ui fo- cz Ui fpsfn 3029- ui fsf fyjtut b)opo–fsp+ fyqpofoujbmgvodujpo jo V0 Opx - fyqpofoujbmn fbot ui bu g tbujtffift

$$g)x\ ,\ \ y+?\ g)x+g)y+ \qquad)95+$$

Y f n bz sfgpsn vmbuf frvbujpo)94+ bt gpmmpx t;

$$g)x+)2\ ,\ \ g)y+, \quad g)iy+, \quad g))2\ ,\ \ i+y++?\ 1.$$

Tjodf $g)x+$i? 1- xf hfu

$$)2\ ,\ \ g)y++)2\ ,\ \ g)iy++?\ 1. \qquad)96+$$

Ui vt $g)y+?$ 2 ps $g)iy+?$ 20 Ui fsf gpsf- jg xf qvu $a ?\ x, b ?\ x\ ,\ \ y$- $c ?\ x\ ,\ \ )2\ ,\ \ i+y$- $d ?\ x\ ,\ \ iy$- ui fo ju gpmmpx t ui bu fjui fs $g)b+/g)a+?\ g)c+/g)d+?$ 2 ps $g)d+/g)a+?\ g)c+/g)b+?$ 20 Ui vt ui f wbmvft pg g bu ui f qpjout a, b, c, d bsf fjui fs $g)a+, \ g)a+, \ g)d+, g)d+$ ps $g)a+, g)b+, \ g)b+, \ g)a+$ sftqfdujwfmz0 Ui vt ui f wbmvft pg G bu ui f wfsujdft pg boz voju trvbsf dpoubjofe cz G dbo cf efdpn qptfe joup ux p qbjst pg ui f gpsn $)x, \ x+$0 Y f efdpn qptf $\mathbb{C}^{\subseteq} ?\ \mathbb{C}$ 1 joup ux p qbsut A boe B tvdi ui bu $A ?$ B0 Mfu $h)x+?$ 2 jg $g)x+$ / A- boe $h)x+?$ $\sqrt{2}$ jg $g)x+$ / B0 Ui fo $h; \mathbb{C} \Leftarrow \}2, \ 2|$ i bt ui f qspqfsuz ui bu gps fwfsz dpohsvfou dpqz Q pg ui f voju trvbsf dpoubjofe cz G- ux p pg ui f wbmvft pg h bu ui f wfsujdft pg Q frvbm 2 boe ux p pg ui f wbmvft pg h bu ui f wfsujdft pg Q frvbm 20

Tjodf h hjwft b 3.dpmpsjoh pg ui f tfu $R \rightarrow G$- ju gpmmpx t ui bu ui fsf jt b trvbsf x jui ui f qspqfsuz ui bu bu mfbtu ui sff wfsujdft cfmpoh up R boe i bwf ui f tbn f dpmps0 Mfu ui ftf qpjout cf a, b, c, d- x i fsf a, b, c / R0 Tjodf $d ?\ \circ a \circ\ b \circ\ c$ / G- ui f trvbsf a, b, c, d cfmpoht up G0

Ui fo- bt x f tbx bcpwf ux p pg ui f wbmvft $h)a+$, $h)b+$, $h)c+$, $h)d+$ frvbm2 boe ux p frvbm 20 Ui jt jt b dpousbejdujpo- tjodf $h)a+?\ h)b+?\ h)c+$. □

Remark 7.10. B tjn jmbs bshvn fou ti px t ui bu ui fsf fwfsz opo.efhfofsbuf qbsbmmfmphsbn i bt ui f ejtdsfuf Qpn qfjv(t qspqfsuz x jui sftqfdu up dpohsvfodft pg ℂ0 Y f offe up vtf ui f tfdpoe qbsu pg Ui fpsfn 8070

Y f dmptf ui jt tfttjpo x jui b cbtjd pqfo rvftujpo- x i jdi dpvme cf ui f bobmphvf pg ui f sftvmu pg Cspx o- Tdi sfjcfs boe Ubzmps]4‘0

Question 7.11. *Is that true that every nonempty finite D has the discrete Pompeiu property with respect the congruences of the plane?*

8 Summary

Jo ui f ui ftjt x f bsf dpodfsofe xjui ui f rjofbs gvodujpobmfr vbujpo

$$\prod_{i=1}^{n} a_i f)b_i x , \ \ c_i y+? \ 1 \qquad)x, y \ / \ \mathbb{C}+ \qquad)97+$$

x i fsf a_i, b_i, c_i bsf hjwfo dpn qmfy ovn cfst- boe f ; $\mathbb{C} \Leftarrow \mathbb{C}$ jt ui f vol opx o gvodujpo0

Cz b x fmml opx o sftvmu pg MDT–fl fmi jej]47‘- voefs tpn f n jme dpoejujpot po ui f frvb. ujpo- fwfsz tpmvujpo pg frvbujpo)97+jt b hfofsbmj{fe qpmzopn jbm0 Cvu ui f ffiofs tusvduvsf pg ui f tpmvujpot i bt cffo jowftujhbufe pomz sfdfoumz0

Jo Tfdujpo 3 x f efffiof tpn f cbtjd opubujpo x i jdi x f vtf ui spvhi pvu ui f ui ftjt0 Y f jouspevdf tpn f hfofsbmj{bujpot pg ui f opujpo pg qpmzopn jbmmz bqqmzjoh gspn bo Bcfmjbo hspvq G up $\mathbb{C}$ boe pvumjof ui f sfmbujpot cfux ffo ui fn 0 Y f dpmmfdu ui f n bjo sftvmut pg ui f ui fpsz pg tqfdusbmbobmztjt boe tzoui ftjt po ejtdsfuf hspvqt po x i jdi ui jt x psl jt cbtfe0 Gjobmmz x f tvn n bsj{f tpn f jn qpsubou sftvmut po rjofbs gvodujpobmfr vbujpot x i jdi tuboet jo pvs dfoufs pg joufsftu0

Ui f n bjo sftvmu pg Tfdujpo 4 jt ui bu *an equation of form*)97+*has a nonzero generalized polynomial solution of degree k if and only if there are field automorphisms* $\phi_1, \ldots, \phi_k$ *of* $\mathbb{C}$ *such that* $\phi_1 \times\times\times \phi_k$ *is a solution.*)tff Ui fpsfn 408- Ui fpsfn 4029- Dpspmmbsz 402: +0 Ui jt sftvmu qspwjeft b ui fpsfujdbmqpttjcjmjuz up efdjef ui f fyjtufodf pg opo.dpotubou tpmvujpot pg frvbujpo)97+x i jdi tbujtffift b x fbl dpoejujpo po ui f qbsbn fufst a_i, b_i, c_i0 Ui f qsppg vtft sftvmut pg tqfdusbmbobmztjt po ejtdsfuf Bcfmjbo hspvqt]31‘)tff Ui fpsfn 3029+0

Ui f jefb pg bqqmzjoh tqfdusbmbobmztjt up ui ftf wbsjfujft po $K^{\subseteq}$? $\}x \ / \ K$; x $\natural$? $1|$ boe $)K^{\subseteq}+^k$ x bt jouspevdfe jo]26‘)i fsf K efopuft b tvcffifme pg $\mathbb{C}$+ x i fsf ju x bt ti px o ui bu)97+*has a nonzero additive solution if and only if there is a solution which is a field automorphism of* $\mathbb{C}$. Jo tpn f tqfdjbmdbtft ui jt x bt qspwfe fbsmjfs cz B0 Wbshb boe Dt0 Wjodf)]53‘-]52‘+0 Ui fjs sftvmut- boe bmtp ui f fbsmjfs ui fpsfn t pg [0 Ebsǒd{z)]8‘-]53‘-]52‘+jojujbufe ui f jefb pg dpoofdujoh ui f fyjtufodf pg opo{fsp beejujwf tpmvujpot up ffifme jtpn psqi jtn t0

Jo Tfdujpo 5 x f ti px ui bu *if the parameters* b_i, c_i *are algebraic, then the products* $\phi_1 \times\times\times \phi_k$ *span the linear space of solutions of degree* k0)Tff bmtp Ui fpsfn t 504 boe 5050+ Jo Tvctfdujpo 504 x f efbmx jui ui f tqfdjbmdbtf x i fo fwfsz beejujwf gvodujpo jt ui f tpmvujpo pg)97+0 Y f dbmmtvdi bo frvbujpo *trivial*0 Ju jt qspwfe ui bu ui f tqbdf pg beejujwf tpmvujpot pg b usjwjbmgvodujpobmfr vbujpo po $\mathbb{C}$ jt tqboofe cz ui f bvupn psqi jtn t pg $\mathbb{C}$)Ui fpsfn 507+0 Ju jt ti px o- vtjoh efsjwbujpot- ui bu ui f beejujwf tpmvujpot bsf opu tqboofe cz bvupn psqi jtn t jo hfofsbm)Ui fpsfn 508+0 Ui ftf sftvmut bsf cbtfe po ui f bsujdmf]24‘0

Jo Tfdujpot 6 boe 7 pvs bjn jt up qsftfou b efotf tvctfu pg tpmvujpot tqbdf S dpotjtujoh pg gvodujpot pg tjn qmf tusvduvsf0

B csjfg gpsn vmbujpo pg pvs n bjo sftvmu jt ui f gpmmpxjoh0 Mfu)97+ cf bo bscjusbsz frvbujpo0 Ui fo *the set of additive solutions defined on K is spanned by those solutions which can be written in the form $\phi \equiv D$, where ϕ is a field automorphism of $\mathbb{C}$ and D is a differential operator on K*)tff Ui fpsfn 60 +0

The set of solutions which are generalized monomials of order k is spanned by those solutions which can be represented as finite sums of functions of the form $\bigcup_{i=1}^{k})\phi_i \equiv D_i+$, where $\phi_1, \ldots, \phi_k$ are field automorphisms of $\mathbb{C}$ and $D_1, \ldots, D_k$ are differential operators on K)tff Ui fpsfn 706+0

If the equation)97+ *is normal of degree k, then the set S is spanned by those solutions which can be represented as finite sums of functions of the form $\bigcup_{i=1}^{m})\phi_i \equiv D_i+$, where $m \geq k$, and ϕ_i and D_i are as above*)tff Dpspmmbsz 708+0

Ui f qsppg pg ui ftf sftvmut jt cbtfe po ui f gbdu ui bu tqfdusbmtzoui ftjt i pmet jo tpn f sfmbufe wbsjfujft)tff Ui fpsfn t 609 boe 704+0 Ui ftf wbsjfujft bsf efgjofe po ui f hspvqt $K^{\subseteq}$ boe- n psf hfofsbmmz- po)$K^{\subseteq}$+^{k}0 Ui ftf hspvqt dpoubjo gsff Bcfmjbo hspvqt pg sbol jogjojuz)tff ui f sfn bsl bgufs Ui fpsfn 605+ boe ju jt xfmm lopxo ui bu po tvdi b hspvq ui fsf bsf wbsjfujft jo xi jdi tqfdusbmtzoui ftjt epft opu i pme)Ui fpsfn 302: +0 Ui jt n fbot ui bu jo psefs up qspwf Ui fpsfn t 609 boe 704 xf i bwf up vtf tpn f tqfdjbmqspqfsujft pg ui f wbsjfujft0 Ui f dsvdjbmpctfswbujpo jt ui bu jo ui ftf wbsjfujft fwfsz mpdbmmz qpmzopn jbm fyqpofoujbmgvodujpo jt b qpmzopn jbmfyqpofoujbmgvodujpo)tff Ui fpsfn t 608 boe 702+0 Ui fo- vtjoh b hfofsbmui fpsfn tubujoh ui bu mpdbmtqfdusbmtzoui ftjt i pmet po fwfsz dpvoubcmf Bcfmjbo hspvq]29‘ xf jogfs ui bu tqfdusbmtzoui ftjt i pmet jo ui ftf wbsjfujft0

Jo Tvctfdujpot 604 boe 704 xf hjwf tfwfsbmbqqmjdbujpot pg ui f hfofsbmui fpsfn t dpo. dfsojoh ui f tpmvujpot pg)97+0

Gjobmmz jo Tfdujpo 8 xf joupspevdf b qspcmfn xi jdi jt dbmmfe *discrete Pompeiu problem*0 Yf tbz ui bu ui f gjojuf tfu $D \to \mathbb{R}^2$ i bt ui f *discrete Pompeiu property with respect to the group G of transformations* jg gps fwfsz gvodujpo f ; $\mathbb{R}^2 \Leftarrow \mathbb{C}$ ui f frvbujpo

$$\prod_{d \emptyset \sigma(D)} f)d+? \ 1$$

gps bmm σ / G jn qmjft $f \subseteq 1$0

Yf vtf mjofbs gvodujpobmfrvbujpot up ti px ui bu fwfsz gjojuf tfu i bt ui f ejtdsfuf Qpn . qfjv qspqfsuz xjui sftqfdu up tjn jmbsjujft pg ui f $\mathbb{R}^2$)tff Qspqptjujpo 806+0 Yf bmtp ti px jo Ui fpsfn 80 ui bu ui f voju trvbsf i bt ui f ejtdsfuf Qpn qfjv qspqfsuz xjui sftqfdu up dpohsvfodft pg $\mathbb{R}^2$0 Ui jt sftvmu jt cbtfe po bo voqvcmjti fe xpsl pg N0 Mbd.lpwjdi boe ui f bvui ps0 Ui f hfofsbmqspcmfn xi fui fs ps opu fwfsz gjojuf tfu i bt ui f ejtdsfuf Qpn qfjv qspqfsuz xjui sftqfdu up dpohsvfodft sfn bjot pqfo0

9 Összefoglaló

B ejtt–fsubdjjcbo b lpwfulf–p uˆqvtv- vhzofwf–fuu rjofbsjt gvhhwfozfhzforfufllfmgphrbm lp–vol ;

$$\prod_{i=1}^{n} a_i f)b_i x\ ,\ \ c_i y+?\ 1 \qquad)x,y\ /\ \mathbb{C}+ \qquad)98+$$

bi pm a_i, b_i, c_i bepuulpn qrfiy t–bn pl - ft f ; $\mathbb{C} \Leftarrow \mathbb{C}$ jtn fsfurfo gvhhwfoz0

T–flfmi jej M0]47' fhz lp–jtn fsufsfen fozf t–fsjou- cj–pozpt fozi f gfmfufrfl n fmfuu n joefo n fhpmbt bmbrhopt`upuu qprjopn 0 Ăn b n fhpmbtpl ffopn bcc t–fslf–fufudtbl b–vupccjfwfl cfo lf–eufl wj–thbmj0

B 30 gfkf–fucfo ofi boz brhqwfup gphbm bu wf–fuvol cf- n fmflfu l vrdtt–fsfqi f– kvu obl b– fhft– ejtt–fsubdjp tpsbo0 Frpt–ps ofi boz rfii futfhft efffo`djpun vubuvol cf- n fmfl fhz Bcfmdtpqpsuspmb lpn qrfiy t–bn pl cb n fop qprjopn gphbm bu lbsklbl lpscf- n ble ptt–fgphrbrkvl b– f–fllfml bqdtprbupt fsfen fozflfu0 Vubob b n vol bol tpsbo brhqwfup t–. fsfqfu lbut–p ejt–lsfutqfl usbrbobrn–jt ft .t–jouf–jt fm frfufujtn fsufukvl 0 Wfhvmb rjofbsjt gvhhwfozfhzforfufl fm frfufofl brhqwfup fsfen fozfjun vubukvl cf- n fmfl blftpccj wj–t. hbrbupl lp–qpouj ubshzbulfq–jl 0

B 40 gfkf–fugp fsfen fozflfoucfcj–poz`ukvl - i phz *egy)98+típusú egyenletnek pontosan akkor van nem azonosan nulla k-ad fokú általánosított polinom megoldása, ha léteznek olyan $\phi_1,\ldots,\phi_k$ testautomorfizmusai a komplex számoknak, melyekre a $\phi_1 \times\!\times\!\times \phi_k$ szorzat megoldás.*)Mbte Uifpsfn 408- Uifpsfn 4029- Dpsprmbsz 402: 0+

F– b– fsfen foz rfihbrbccjt fm frfujrfih rfii fuptfhfu be bssb- i phz frmpoutvl fhz)98+ uˆqvtv- ft b– a_i, b_i ft c_i qbsbn fufsflsf wpobul p–p fozi f gfmfufrmfl)rbte b–)6+gfmfufm+ frfihfu ufwp fhzforfuspmi phz n jl ps rfuf–jl ofn ovmb n fhpmbtb0 B cj–poz`ubt b ejt–lsfu tqfl usbrbobrn–jt fsfen fozfjo brbqt–jl]31')rbte Uifpsfn 3029+0

B– purfu- i phz tqfl usbrbobrn–jtu brhbm b–i buvol cj–pozpt- b $K^{\subseteq}$? $\}x\ /\ K$; x ?̸ 1| n vmjqrjl bu`w dtpqpsupo- jmfuwf b)$K^{\subseteq}$/k.o efffojbrm wbsjfubtplsb)juu K b $\mathbb{C}$ fhz sft–uftuf+ b]26' eprhp–buc pmt–bsn b–jl 0 Juu n fhn vubuuvl - i phz *a)98+nek akkor és csak akkor van nem-nulla additív megoldása, ha létezik olyan testautomorfizmusa a komplex számoknak, amely maga is megoldás.* Cj–pozpt tqfdjbrjt ftfufl sf b lfseftu Wbshb B0 ft Wjod–f Dt0 n fhwbrbt–pmbl)]53'-]52'+0 Ebspd–z [0 ft b– frpcc fn mfuuu lvubupl fsfen fozfj)]8'-]53'-]52'+ wfufufl gfmb ofn .ovmb beeju`w n fhpmbtpl ft b uftuj–pn psffi–n vtpl lp–puuj lbqdtprbupu0

B 50 gfkf–fucfo n fhn vubukvl - i phz *amennyiben a b_i, c_i paraméterek algebraiak $\mathbb{Q}$ felett, akkor a $\phi_1 \times\!\times\!\times \phi_k$ alakú szorzatok kifeszítik a K-n értelmezett, pontosan k-ad fokú megoldások lineáris terét*0)Mbte Uifpsfn 504 ft Uifpsfn 5050+ B 504 brgfkf–fucfo pmbo gvhhwfozfhzforfufllfmgphrbrhp–vol - n fmfl ofl n joefo beeju`w gvhhwfoz n fhpmbtb0 F–f.

lfu ofwf—vl *triviális* fhzforfufl ofl0 Cfrbukvl- i phz fhz usjwjbrjt gvhhwfozfhzforfu be. ejuˇw n fhprnbtbjobl ufsfcfo tvsvfl b lpn qrfy t–bn pl bvupn psffi-n vtbj0)Juu b tvsvtfh gphbm bi p– b t–ps–buupqprphjbu i bt–obrkvl 0+ Vhzbobl lps b efsjwbdjpl tfhˇutfhfwfmn v. ubuvol qfrnbu bssb- i phz b lpn qrfy t–bn pl obl b–po bvupn psffi-n vtbj- n fmfl fhz bepuu fhzforfun fhprnbti bm b–bc b ftofl- ofn gfmfurfovnuf t–ˇujl l j b– beejuˇwn fhprnbtpl ufsfu0 F–fo fsfen fozfl b]24' djl lful pwfujl0

B 60ft 70gfkf–fufl cfo gp dfmuol- i phz n fhbekvl b n fhprnbtpl i bm b–bobl fhz pmbo tvsv sft–i bm b–bu- bn fm fhzt–fsv t–fsl f–fuv gvhhwfozfl cpmbmu

Spwjefo ptt–fgphrbmb b– fsfen fozfl b lpwful f–pl0 Wfhzvol fhz ufut–prfihft)98+ uˇqvtv fhzforfuf u0 Fl lps *a K-n értelmezett additív megoldások terét kifeszítik azon megoldások, melyek felírhatóak $\phi \equiv D$ alakban, ahol ϕ egy testautomorfizmusa a $\mathbb{C}$-nek, mely maga is egy megoldás és D egy differenciáloperátor a K-n*)rbte Ui fpsfn 60 +0

A pontosan k-ad fokú megoldások terét kifeszítik azon megoldások, amelyek előállnak mint $\bigcup_{i=1}^{k})\phi_i \equiv D_i+$ alakú szorzatok véges összegei, ahol $\phi_1, \ldots, \phi_k$-k a $\mathbb{C}$ testautomorfizmusai és a $\bigcup_{i=1}^{k} \phi_i$ szorzat maga is megoldás, valamint $D_1, \ldots, D_k$ differenciáloperátorok K-n)rbte Ui fpsfn 706+0

Ha a)98+ típusú egyenlet normális (azaz minden megoldása általánosított polinom), melyben a megoldások foka legfeljebb k, akkor a megoldások halmazát kifeszítik azon megoldások, melyek felírhatóak $\bigcup_{i=1}^{m})\phi_i \equiv D_i+$ alakban, ahol $m \geq k$, és ϕ_i illetve D_i mint fent)rbte Dpspmbsz 708+0

B cj–pozˇubt b–po brbqt–jl- i phz b n fhgfrfmp wbsjfubtpl po ufrkftvmb tqfl usbrm–jouf–jt)rbte Ui fpsfn 609 ft Ui fpsfn 704+0 B wj–thbm wbsjfubtpl b $K^{\subseteq}$ n vmjqmjl buˇw dtpqps. upo- jmfuwf bmbbrhoptbccbo- b)$K^{\subseteq}$+.o wboobl efffbojbmb0 F–fo dtpqpsupl ubsubm b–obl wfhufmfo sbohv t–bcbe Bcfmdtpqpsupu0 Jtn fsfuft- i phz jmfo dtpqpsupl po wboobl pmbo wb. sjfubtpl- n fmfl sf b tqfl usbrm–jouf–jt ofn ufrkftvm)rbte Ui fpsfn 302: +0 F– ftfuvol cfo b–u fsfen fozf–j- i phz b gfoufn mfuu ufufmfl cj–pozˇubtbi p– b wbsjfubtbjol cj–pozpt tqfdjbrjt uvmbkepotbhbjul fmf ji bt–obmvol0 B epoup ft–sfwfufmb l pwful f–p0 B– bmbmuol wj–thbm wb. sjfubtpl cbo n joefo mpl brjt qprjopn .fyqpofodjbrjt gvhhwfoz wbmpkbcbo qprjopn .fyqpofodjbrjt)rbte Ui fpsfn 608 ft Ui fpsfn 702+0 Fl lps b mpl brjt qprjopn .fyqpofodjbrjt gvhhwfozfl sf wpobul p–p tqfl usbrfm frfufubmbm b–wb cfrbui bukvl- i phz b 'l p–potfhft' tqfl usbrm–jouf–jt jt ufrkftvmf–fl cfo b wbsjfubtpl cbo0

B– 604 ft b 704 brgfkf–fufl cfo ofi boz brbm b–btun vubvol b)98+uˇqvtv fhzforfufl lfm lbqdtprbupt bmbbrhopt fsfen fozfl sf0

Wfhvmb 80 gfkf–fucfo cfn vubukvl b– vhzofwf–fuu *diszkrét Pompeiu-problémát*0 Fhz $D \to \mathbb{R}^2$ wfhft i bm b–sb b–un poekvl- i phz sfoefrhf–jl b *diszkrét Pompeiu-tulajdonsággal a G transzformációcsoportra nézve,* i b b– $f \subseteq 1$ b– fhzfufo pmbo $\mathbb{R}^2 \Leftarrow \mathbb{C}$ gvhhwfoz-n fmsf

$$\prod_{d\emptyset\sigma(D)} f)d+? \;1$$

gfoobmm joefo σ / G.sf0

Mjofbsjt gyhhwfozfhzforfufl tfh"utfhfwfmn fhn vubukvl - i phz n joefo wfhft D i bm b–sfoefrhf–jl b ejt–l sfu Qpn qfjv.uvrhkepotbhhbmb t"l i btpomptbhj usbot–gpsn bdjpjsb of–wf)rbte Qspqptjujpo 806+0 N btsft–u- b rfozfhftfo ofi f–fcc ftfufu- bn jl ps G b t"l fhz. cfwbhptbhbjobl dtpqpsukb- cj–pozpt tqfdjbrjt ftfucfo t–joufo l f–fmj uvekvl 0 N fhn vubukvl - i phz i b D fhz ofn frgbkvrp qbsbmfurphsbn n b dtvdti bm b–b- bl l ps D sfoefrhf–jl b ejt–l sfu Qpn qfjv.uvrhkepotbhhbmb t"l fhzcfwbhptbhbjsb of–wf)rbte Ui fpsfn 80 +0 F–fo ufufmfhz Mbd–l pwjdi N jl rpttbml p–pt qvcrjl brbubo fsfen fozfo brbqt–jl 0 Ozjupuu l fseft- i phz n jo. efo wfhft i bm b– sfoefrhf–jl .f b ejt–l sfu Qpn qfjv.uvrhkepotbhhbmb t"l fhzcfwbhptbhbjsb of–wf0

References

]2‘ K0B0Cblfs- B hfofsbmgvodujpobmfrvbujpo boe jut tubcjmjuz- *Proc. Am. Math. Soc.* **133**)3115+ op07- 2768.27750

]3‘ L0Cbspo boe Y0Kbsd–zl- Sfdfousftvmut po gvodujpobmfrvbujpot jo b tjohmf wbsjbcmf- qfstqfdujwft boe pqfo qspcmfnt- *Aequationes Math.* **61**)3112+ 2.590

]4‘ M0Cspxo- C0N0Tdisfjcfs boe C0B0Ubzmps- Tqfdusbmtzoui ftjt boe uif qpnqfjv qspcmfn- *Ann. Inst. Fourier* **23**)2:84+ 236.2650

]5‘ Dibrbmpw- Tvs vo qspcmfnf ef E0Qpnqfjv- *Annaire Univ. Sofia Fac. Phys. Math.*- Mjwsf 2- **40**)2:55+ 2.550

]6‘ D0Disjtupw- Tvs vo qspcmfnf ef N0Qpnqfjv- *Mathematica(Timisoara)* **23**)2:58.59+ 214.2190

]7‘ D0Disjtupw- Tvs m(frvbujpo joufhsbmf hfofsbmjtff ef N0Qpnqfjv- *Annaire Univ. Sofia Fac. Phys. Math.*- Mjwsf 2- **45**)2:59.5:+ 278.2890

]8‘ [0 Ebsǒd–z- Opuxfoejhf voe ijosfjdifoef Cfejohvohfo gvs ejf Fyjtufo– wpo ojdiu lpotuboufo Mptvohfo mjofbsfs Gvodujpobmhmfjdivohfo- *Acta. Sci. Math.*)T–fhfe+ **22**)2:72+ 42.520

]9‘ E0Ž0Eplpwjd- B sfqsftfoubujpo uifpsfn gps)X_1 2+)X_2 2+. . .)X_n 2+boe jut bqqmjdbujpot- *Ann. Polon. Math.* **22**)2:7:+ 29:"2:90

]:‘ S0K0Fmmjpu- Uxp opuft po tqfdusbmtzoui ftjt gps ejtdsfuf Bcfmjbo hspvqt- *Proc. Camb. Phil. Soc.* **61**)2:76+ 728.7310

]21‘ N0Fohfsu- Gjojuf ejnfotjpobmusbotmbujpo jowbsjbou tvctqbdft- *Pacific J. Math.*- **32**)2:81+ 444.4540

]22‘ O0Hbspgbmp- Uif ofx sftvmu po uif Qpnqfjv qspcmfn *Rend. Sem. Mat. Univ. Pol. (Torino)*)2:99+ 36.490

]23‘ E0J0Hvsfwjd- Dpvoufsfybnqmf up b qspcmfn pg M0Tdixbsu–- *Funkcional. Anal. i Priložen* **9**)2:86+ op03- 3:.460
)Fohmjti usbotmbujpo; *Funckcional Anal. Appl.* **9**)2:86+ op03- 227.2310+

]24‘ H0Ljtt- Mjofbs gvodujpobmfrvbujpot xjui bmhfcsbjd qbsbnfufst- bddfqufe gps qvcmjdbujpo jo Qvcmjdbujpoft Nbuifnbujdbf Efcsfdfo0

]25‘ H0Ljtt boe N0Mbd-l pwjdi - Mjofbs gvodujpobmfr vbujpot- ejfifsfoujbmpqfsbupst boe tqfdusbmtzoui ftjt- bddfquf e gps qvcrjdbujpo jo Bfr vbujpoft N bui 0

]26‘ H0Ljtt boe B0Wbshb- Fyjtufodf pgopoustjwjbmtpmvujpot pgrjofbs gvodujpobmfr vbujpo-qvcrjti fe porjof *Aequationes Math.*)3124+0

]27‘ N0Lvd-n b; *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality.* Qbǒtux px f Y zebx ojdux p Obvl px f-Vojx 0Ťhbtl j- Y bst–bx b.Lsbl ǒx .L bupx jdf- 2: 960 Tf dpoe fejujpo; Cjsl i bvtfs- 311:)B0Hjrhozj fejups+0

]28‘ N0 Mbd-l pwjdi - Qpmzopn jbmn bqqjoht po Bcfrjbo hspvqt- *Aequationes Math.* **68**)3115+ 288.2: : 0

]29‘ N0Mbd-l pwjdi - Mpdbmtqf dusbmtzoui ftjt po Bcfrjbo hspvqt- up bqqf bs jo *Acta Math. Hungar.*

]2: ‘ N0Mbd-l pwjdi - Qprjopn .fyqpofodjbrjt gvhhwf ozfl - *Matematikai Lapok* **10**)2+)3111.3112+ 33.420

]31‘ N0 Mbd-l pwjdi boe H0 T-fl fmi jej- I bsn pojd bobmztjt po ejdsfuf bcfrjfo hspvqt-*Proc. Am. Math. Soc.* **133**)3115+ op0 7- 2692.26970

]32‘ N0 Mbd-l pwjdi boe M0 T-fl fmi jej- Tqf dusbmtzoui ftjt po ejdsfuf Bcfrjbo hspvqt-*Math. Proc. Camb. Phil. Soc.* **143**)3118+ 214.2310

]33‘ N0Mfgsbod- Bobmztf tqf dusbrf tvs $\mathbb{Z}^n$- *C. R. Paris* **246**)2: 69+ 2: 62.2: 64

]34‘ T0N b–vs boe Y 0P srjd–- Hsvoeml hf oef Fjhf otdi bguf o efs qpmzopn jtdi fo Pqfsbujp-ofo- *Fund. Math.*)2: 4: + 53.610

]35‘ [t0Qbrfit- Fyuf otjpo ui fpsfn t gpfs gvodujpobmfr vbujpot x jui cjtzn n fusjd pqfsbupst-*Aequationes Math.* **63**)3113+ 377.3: 20

]36‘ E0Qpn qf jv- Tvs vof qspqsjfuf eft gpodujpot dpoujovft efqf oefouef qmtjf vst wbsj-bcrft *Bull. Sci. Math(2)* **53**)2: 3: + 439.4430

]37‘ N0K0Qvm- Ui f Qpn qf jv qspcrfn boe ejtdsfuf hspvqt- bddfquf e- *Monatsh. Math.* **172** op0 4.5)3124+ 526.53: 0

]38‘ B0H0S bn n - Ui f Qpn qf jv qspcrfn - *Applicable Analysis* **64** op0 2.3)2: : 8+ 2: .370

]39‘ S0S bep- Byjpn bujd usf bun fou pg sbol jo joffojuf tfut- *Canad. J. Math* **1**)2: 5: + 448.4540

]3: ‘ T0Tuf wjč- Cfi bwjpvs pgtpmvujpot pgtpn f rjofbs gvodujpobmfr vbujpot bujoffojuz- *Appl. Math. Comput.* **219**)3124+ 7245.72520

]41‘ M0F0Ti befs- Bmmsjhi u usjbohmft bsf Sbn tfz jo E^{2}"- *Journ. Comb. Theory (A)* **20**)2: 87+ 496.49: 0

]42‘ M0T-flf mai jej- Sfn bsl po b qbqfs pg N0 B0 N dLjfsobo- *Ann. Polon. Math.* **36**)2: 8: + 356.3580

]43‘ M0T-flf mai jej- Opuf po fyqpofoujbmqpmzopn jbmt- *Pacific J. Math.* **103**)2: 93+ 694. 6980

]44‘ M0T-flf mai jej- Po b dmbtt pgrjofbs gvodujpobmfr vbujpot - *Publ. Math.*)Efcsfdfo+**29**)2: 93+ 2: .390

]45‘ M0T-flf mai jej- Opuf po fyqpofoujbmqpmzopn jbmt- *Pacific J. Math.* **103**)2: 93+ 694. 6980

]46‘ M0T-flf mai jej- ui f gbjmvsf pg tqfdusbmtzoui ftjt po tpn f uzqft pg ejtdsfuf Bcfmjbo hspvqt- *J. Math. Anal. Appl.* **291**)3115+ 868.8740

]47‘ M0T-flf mai jej- *Discrete Spectral Synthesis and Its Applications*- Tqsjohfs.Wfsmbh- 31170

]48‘ N0Usvt-d-zotl j- [t0Uv-b- Sbep(t Tfmfdujpo Qsjodjqmf; bqqmjdbujpot up cjobsz sfmb. ujpot- hsbqi boe i zqfshsbqi dpmpsjoht boe qbsujbmmz psefsfe tfut- *Discrete Mathematics* **103**)2: : 3+ 412.4230

]49‘ Wjod-fof Wbshb Besjfoo; *Középértékeket tartalmazó függvényegyenletek.* Qi E ui ftjt- Efcsfdfo- 31230

]4: ‘ B0 Wbshb- Po b gvodujpobmfr vbujpo dpoubjojoh gpvs xfjhi ufe bsjui n fujd n fbot- *Banach J. Math. Anal.* **2**)2+)3119+ 32.430

]51‘ B0Wbshb- Po beejujwf tpmvujpot pgb rjofbs fr vbujpo- *Acta Math. Hungar.* **128**)2.3+)3121+ 26.360

]52‘ B0 Wbshb- Dt0 Wjod-f- Po Ebsp̌d-z(t qspcmfn gps beejujwf gvodujpot- *Publ. Math.*)Efcsfdfo+**75**)2.3+)311: + 3: : .4210

]53‘ B0 Wbshb- Dt0 Wjod-f- Po b gvodujpobmfr vbujpot dpoubjojoh xfjhi ufe bsjui n fujd n fbot- *Inter. Ser. Num. Math.* **157**)311: + 416.4260

]54‘ T0Y jmjbn t- B qbsujbmtpmvujpot pg Qpn qfjv qspcmfn - *Math. Ann.* **223**)2: 87+ 294. 2: 10

]55‘ T0Y jmjbn t- Bobmzujdjuz pg ui f cpvoebsz gps Mjqtdi ju- epn bjot xjui pvu Qpn qfjv qspqfsuz- *Indiana Univ. Math. Journ.***30**)2: 92+468.47: 0

]56‘ Wbo efs Y bfsefo- *Algebra*- Gsfefsjdl Vohbs Qvcmjtijoh Dp0 Ofx Zpsl - 2: 81- Wpm0 30

]57‘ M0[bmdn bo- B cjcmjphsbqi jdbmtvswfz pg ui f Qpn qfjv Qspcmfn - jo ui f cppl *Approximation by solutions of partial differential equations.* Fejufe cz C0Gvhmfef0Lmvxfs Bdbe0 Epsesfdi u- 2: : 3- 288.2970

]58‘ E0[fjmfshfs- Qpn qfjv(t qspcmfn jo ejtdsfuf tqbdf- *Proc. Nat. Acad. Sci. USA* **75**)2: 89+ op0 9- 4666.46670